Viernes 29 · enero · 2010.

Let $x \geq 0$. $\Rightarrow$ * $1 - e^{-x}$ increasing

* $1 - e^{-x} \leq 1$

* $1 - e^{-x} \leq x$

$\hookleftarrow$ $e^{-x} \leq 1$

$\Rightarrow 1 - e^{-x} \geq 0$

$\Rightarrow 0 \leq \int_0^x (1 - e^{-y}) dy$

$= x - (1 - e^{-x})$

$\Rightarrow x \leq 1 - e^{-x}$

* $\frac{1 - e^{-x}}{x} \xrightarrow[x \to 0^+]{} 1$

$\hookrightarrow$ $(-e^{-x})'|_0 = \lim_{x \to 0} \frac{(-e^{-x}) - (-e^{-0})}{x - 0}$

$= \lim_{x \to 0} \frac{1 - e^{-x}}{x}$

But $(-e^{-x})'(0) = [+ e^{-x}]|_{x=0} = 1$

* $\frac{1 - e^{-x}}{x}$ decreasing.

* $\int_A (1-e^{-f})\,d\mu \le \int_A 1\,d\mu = \mu(A)$

and $\int_A (1-e^{-f})\,d\mu \le \int_A f\,d\mu$

Now, we define $\frac{1-e^{-f}}{f}$ as 1 for $f(x)=0$.

$\Rightarrow \int_A (1-e^{-f})\,d\mu = \int_A f\,\frac{1-e^{-f}}{f}\,d\mu$

$$\ge \inf_A \left(\frac{1-e^{-f}}{f}\right) \int_A f\,d\mu$$

Also, $\int_A (1-e^{-f})\,d\mu \ge \int_A \inf_A (1-e^{-f})\,d\mu$

$$\ge \inf_A (1-e^{-f})\,\mu(A)$$

Let's see that $\inf_A (1-e^{-f}) = 1-e^{-\inf_A f}$

$$\inf_A \frac{(1-e^{-f})}{f} = \frac{1-e^{-\sup_A f}}{\sup_A f}$$

$$\Rightarrow \left.\begin{matrix} 1-e^{-\inf_A f} \\ \frac{1-e^{-\sup_A f}}{\sup_A f} \end{matrix}\right\} \le \int_A (1-e^{-f})\,d\mu \le \left\{\begin{matrix} \mu(A) \\ \int_A f\,d\mu. \end{matrix}\right.$$

Notation $\rightarrow$ $\{f\le\varepsilon\} = f^{-1}([0,\varepsilon]) = \{x \mid f(x)\le\varepsilon\},\ x\in G.$

* $\int_{\{f\le\varepsilon\}} (1-e^{-f})\,d\mu \le \int_{\{f\le\varepsilon\}} f\,d\mu$

On the other hand,

$$\int_{\{f\le\varepsilon\}} f\frac{(1-e^{-f})}{f}\,d\mu \ge \frac{1-e^{-\varepsilon}}{\varepsilon}\int f\,d\mu.$$

* $\int_{\{f\le\varepsilon\}} (1-e^{-f})\,d\mu < \infty \iff \int_{\{f<\varepsilon\}} f\,d\mu < \infty.$

$\int_{\{f>\varepsilon\}} (1-e^{-f})\,d\mu < \infty \iff \mu(\{f>\varepsilon\}) < \infty$

* $\int_{\{f>\varepsilon\}} (1-e^{-f})\,d\mu \le \mu(\{f>\varepsilon\})$

Also, $\int_{\{f>\varepsilon\}} (1-e^{-f})\,d\mu \ge (1-e^{-\varepsilon})\,\mu(\{f>\varepsilon\})$

Conclusion

$\int (1-e^{-f})\,d\mu < \infty \iff \exists\, \varepsilon>0$ t. $\int_{\{f\le\varepsilon\}} f\,d\mu < \infty$ and $\mu(\{f>\varepsilon\}) < \infty$.

Then, we will have that it happens $\forall \varepsilon > 0$.

Theorem → If ① $\int (1-e^{-f})\,d\mu = \infty$,

$$\Rightarrow \mathbb{P}\{\int f\,dN = \infty\} = 1$$

and, if ② $\int (1-e^{-f})\,d\mu < \infty$,

$$\Rightarrow \mathbb{P}\{\int f\,dN = \infty\} = 0.$$

Proof → ① $\mathbb{E}\{e^{-\int f\,dN}\} = e^{-\int (1-e^{-f})\,d\mu}$

$= 0$ if $\int (1-e^{-f})\,d\mu = \infty$

If $\mathbb{P}\{X \geq 0\}$ with $\mathbb{E}(X) = 0 \iff \mathbb{P}\{X=0\} = 1$.

$X = e^{-\int f\,dN}$

$X = 0 \iff \int f\,dN = \infty$.

$\mathbb{E}\{e^{-\int f\,dN}\} = 0 \Rightarrow \mathbb{P}\{e^{-\int f\,dN} = 0\} = 1$

$\iff \mathbb{P}\{\int f\,dN = \infty\} = 1$.

Take $x \geq 0$, $\theta > 0$.

$\mathbb{E}(e^{-\theta X}) = \mathbb{E}\{e^{-\theta X}\, \mathbb{I}_{\{X<\infty\}}\} \xrightarrow[\theta \downarrow 0]{\text{monotone convergence}} \mathbb{E}\{\mathbb{I}_{\{X<\infty\}}\}$

$= \mathbb{P}\{X < \infty\}$

$\Rightarrow \forall x,\ \mathbb{P}(X \geq 0),\ \lim_{\theta \to 0} \mathbb{E}(e^{-\theta X}) = \mathbb{P}\{X < \infty\}$

② $\int (1-e^{-f})\,d\mu < \infty. \Rightarrow \forall \varepsilon > 0\ \int_{\{f \leq \varepsilon\}} f\,d\mu < \infty$,

$\mu(\{f > \varepsilon\}) < \infty$.

$\underbrace{\mathbb{E}\{e^{-\theta \int f\,dN}\}} = e^{-\int (1-e^{-\theta f})\,d\mu} \xrightarrow[\theta \downarrow 0]{} ?$

$\xrightarrow[\theta \downarrow 0]{} \mathbb{P}\{\int f\,dN < \infty\}$

↳ $\int (1-e^{-\theta f})\,d\mu$.

$= \underbrace{\int_{\{f \leq \varepsilon\}} (1-e^{-\theta f})\,d\mu}_{\leq \theta f} + \underbrace{\int_{\{f > \varepsilon\}} (1-e^{-\theta f})\,d\mu}_{\leq 1}$

$\leq \theta \int_{\{f \leq \varepsilon\}} f\,d\mu + \mu(\{f > \varepsilon\})$

$\xrightarrow[\theta \downarrow 0]{} \mu(\{f > \varepsilon\}) \xrightarrow[\varepsilon \to \infty]{f: G \to \mathbb{R}} 0$.

$\Rightarrow \mathbb{P}\{\int f\,dN < \infty\} = \lim_{\theta \downarrow 0} e^{-\int (1-e^{-\theta f})\,d\mu} = e^{-0} = 1$

$\therefore \mathbb{P}\{\int f\,dN = \infty\} = 0$.

Lunes → 9.2.4)

Example

$G = [0,\infty)$, $\mathcal{G} = \mathcal{B}([0,\infty))$, $\mu = \lambda \ell$ (Lebesgue measure), $\lambda > 0$

$A_1 = (0, t_1]$, $A_2 = (t_1, t_2]$, ..., $A_k = (t_{k-1}, t_k]$

$N(A_1), \dots, N(A_k)$ ind.

$N(A_i) \sim \text{Poisson}(\mu(A_i))$
$= \text{Poisson}(\lambda(t_i - t_{i-1}))$

Denote $N_t = N([0,t])$,

$\Rightarrow N_{t+s} - N_s = N(s, s+t) \sim \text{Poisson}(\lambda t)$

$\{N_{t_i} - N_{t_{i-1}} \mid i = 1, \dots, k\}$ independent.

* N_t right continuous

Take $t_n \downarrow t$

$$\lim_{n\to\infty} N_{t_n} = \lim_{n\to\infty} N([0,t_n])$$

$$= N\left(\bigcap_{n=1}^{\infty} [0,t_n]\right) = N([0,t]) = N_t$$

*If N is a Poisson random measure on $([0,\infty), \mathcal{B}([0,\infty)))$ with average measure $\mu(A) = \lambda\, \ell(A)\ \forall A \in \mathcal{B}([0,\infty))$,

$\Rightarrow N_t = N([0,t])$ is a stochastic process that satisfies the following:

1) $N_t \geq 0$, right continuous, with $N_t \in \mathbb{N} \cup \{0\}$, and non-decreasing.

↳ COUNTING PROCESS

2) $N_{s+t} - N_s \sim N_t \sim \text{Poisson}(\lambda t)$

↳ Stationary increments

3) $\forall k \in \mathbb{N}$, $\mathcal{P} = \{0 = t_0 < t_1 < \dots < t_k\}$,
$\{N_{t_i} - N_{t_{i-1}} \mid i = 1, \dots, k\}$ ind. ← independent increments

* $\{N_t \mid t \geq 0\}$ is a **Poisson process with rate λ**

> Result: N_t counting process
> 1) stationary increments
> 2) ind. inc.
> 3) simple
> ↑ saltos de, a lo mucho, 1 en instantes.
>
> $\Rightarrow \{N_t \mid t \geq 0\}$ is a Poisson process

Lunes 1° feb. 2010.

Non homogeneous Poisson Process.

$\{N_t \mid t \geq 0\}$ is called a non-homogeneous Poisson process with rate function $\lambda(t) \geq 0$ (integrable on finite intervals: $\int_s^t \lambda(u)\,du < \infty \ \forall\ 0 \leq s \leq t < \infty$)

if
i) N counting process
ii) N independent increments
iii) $N(t+s) - N(s) \sim \text{Poisson}\left(\int_s^t \lambda(u)\,du\right)$

* for us, a non-homogeneous Poisson process is $N_t = N([0,t])$ when is a Poisson random measure on $([0,\infty), \mathcal{B}([0,\infty)))$ with some σ-finite average measure μ.
↑ Intensity.

* $\mathcal{P} = \{0 = t_0 < t_1 < t_2 < \dots < t_n\}$

$\{N_{t_i} - N_{t_{i-1}} \mid i = 1, \dots, n\}$ ind.

$N([t_{i-1}, t_i])$

A_i ← disjoint.

$N_{t+s} - N_s = N([s, t+s]) \sim \text{Poisson}(\mu((s, s+t]))$

$\mu((s,s+t]) = \int_s^{s+t} \lambda(u)\,du.$

* Assume that $\mu((s, s+t]) = \Lambda(t+s) - \Lambda(s)$, where $\Lambda(\cdot)$ is non-decreasing and continuous. That is, μ has no atoms.

> atom: $\mu(\{x\}) > 0$.
> $\Lambda(t) = \mu([0,t])$

* Take $\tilde{N}_t$ a Poisson process with rate 1.

$\tilde{N}$ is a counting process with independent and stationary increments, $\tilde{N}(t+s) - N(s) \sim \text{Poisson}(t)$.

$N_t = \tilde{N}_{\Lambda(t)}$ is a counting process.

$N_{t_i} - N_{t_{i-1}} = \tilde{N}_{\Lambda(t_i)} - \tilde{N}_{\Lambda(t_{i-1})}$

$(\Lambda(t_{i-1}), \Lambda(t_i)]$ disjoint.

$s_i = \Lambda(t_i)$; $s_1 \le s_2 \le \ldots \le s_n$.

$\{\underbrace{\tilde{N}_{s_i} - \tilde{N}_{s_{i-1}}}_{N_{t_i} - N_{t_{i-1}}} \mid i = 1, \ldots, n\}$ ind.

$N_t - N_s = \tilde{N}_{\Lambda(t)} - \tilde{N}_{\Lambda(s)} \sim \text{Poisson}(\Lambda(t) - \Lambda(s))$

* $X_1, X_2, \ldots$ iid. $\sim F$ continuous.

$\mathbb{P}(X_i = 0) < 1$

$\tau_1 = 1 \qquad Y_1 = X_{\tau_1} = X_1$.

$\tau_2 = \inf\{i > \tau_1 \mid X_i > Y_1\}$

$Y_2 = X_{\tau_2}$.

⋮

$\tau_k = \inf\{i > \tau_{k-1} \mid X_i > Y_{k-1}\}$

$Y_k = X_{\tau_k}$.

⋮

$Y_1 < Y_2 < Y_3 < \ldots$

$N_t = \sup\{i \mid Y_i \le t\}$

↑ How many records have we seen until t?

$\mathbb{P}\{Y_{n+1} > t \mid Y_1 = s_1, \ldots, Y_{n-1} = s_{n-1}, Y_n = s\}$

$= \mathbb{P}\{Y_{n+1} > t \mid Y_n = s\}$

$= \mathbb{P}(X > t \mid X > s) = \dfrac{\mathbb{P}(X > t, X > s)}{\mathbb{P}(X > s)}$

$\stackrel{\text{sup. } t > s}{=} \dfrac{\mathbb{P}\{X > t\}}{\mathbb{P}\{X > s\}}$

$= \bar{F}_X(t) / \bar{F}_X(s)$.

$\mathbb{P}(X > t) = 1 - F_X(t) = \bar{F}_X(t)$

$= 1$ if $t \le s$

* $\mathbb{P}\{Y_1 > t\} = \bar{F}_X(t)$

$\mathbb{P}\{Y_1 \le t < Y_2\} = \int_0^t \mathbb{P}\{Y_2 > t \mid Y_1 = s\}\, dF(s)$

$= \int_0^t \dfrac{\bar{F}_X(t)}{\bar{F}_X(s)}\, dF(s) = \bar{F}(t) \int_0^t \dfrac{dF(s)}{\bar{F}_X(s)}$

If g is non decreasing with finite variation and continuous, $\Rightarrow$ $\forall$ f differentiable.

$$f(g(b)) - f(g(a)) = \int_a^b f'(g(s))\, dg(s)$$

Stieltjes
$$= \int_{g(a)}^{g(b)} f'(x)\, dx$$

$$= \bar{F}_x(t) \int_0^t \frac{1}{1-F_x(s)}\, dF(s) = \bar{F}_x(t) \int_{F_x(0)}^{F_x(t)} \frac{dx}{1-x}$$

$$= \bar{F}_x(t) \left\{ -\ln(1-x) \Big]_{F(0)}^{F(t)} \right\}$$

$$= \bar{F}_x(t) \left\{ \ln(1-F(0)) - \ln(1-F(t)) \right\}.$$

Let's assume $F(0) = 0$.

Denote $\Lambda(t) = \begin{cases} -\ln(1-F_x(t)); & F(t) < 1,\ t \geq 0 \\ \infty; & F(t) = 1;\ t \geq 0 \end{cases}$

$$= \bar{F}_x(t) \{\Lambda(t)\}$$

We define $T = \sup\{t \mid F(t) < 1\}$.

$$\lim_{t \to T} \Lambda(t) = \infty.$$

$$\Lambda(t) < \infty \quad \forall\ t < T.$$

$$\Rightarrow \mathbb{P}\{Y_1 > t\} = \bar{F}(t) = e^{-\Lambda(t)}$$

$0 < t < T$.

$$\mathbb{P}\{Y_1 \leq t < Y_2\} = \Lambda(t)\, \bar{F}_x(t)$$

$$= \Lambda(t)\, e^{-\Lambda(t)}$$

* $\mathbb{P}\{N_t = 0\} = \mathbb{P}\{Y_1 > t\} = e^{-\Lambda(t)}$

$$\mathbb{P}\{N_t = 1\} = \mathbb{P}\{Y_1 \leq t < Y_2\} = \Lambda(t)\, e^{-\Lambda(t)}.$$

$$\mathbb{P}\{N_t = n\} = e^{-\Lambda(t)} \frac{\Lambda^n(t)}{n!}$$

$\leftarrow N_t \sim \text{Poisson}(\Lambda(t))$.

* $dF_{Y_1,\dots,Y_n \mid N_t = n}(s_1,\dots,s_n) = \dfrac{\cancel{e^{-\Lambda(t)}} \prod_{i=1}^{n} d\Lambda(s_i)}{\cancel{e^{-\Lambda(t)}} \frac{\Lambda^n(t)}{n!}}$

$$= n! \prod_{i=1}^{n} \frac{d\Lambda(s_i)}{\Lambda(t)}$$

↳ distribution of the n-th order statistic taken from iid. r.v. Ω_i with distribution

$$\mathbb{P}(\Omega_i \in A) = \frac{\mu(A \cap [0,t])}{\mu([0,t])} \leftarrow \Lambda(t)$$

$$= \frac{\int_0^t \mathbb{I}_A(s)\, d\Lambda(s)}{\Lambda(t)}$$

Therefore $N([a,b]) = N_b - N_a$

↳ N is a Poisson measure with average measure μ.

* For any Λ continuous $\ni$ $\Lambda(0) = 0$, $\Lambda(t) \xrightarrow[t \uparrow T]{} \infty$ for some $T \leq \infty$

we have that

$F(t) = 1 - e^{-\Lambda(t)}$ is a cont. distribution function with $F(0) = 0$, $F(t) \xrightarrow[t \uparrow T]{} 1$.

For any such distribution, take $\Lambda(t) = -\ln \bar{F}_x(t)$

* N_1, N_2 are ind. random measures if $\forall f_1, f_2 \geq 0$ Borel $\int f_1 dN_1 \perp \int f_2 dN_2$.

* $N_1, N_2, \dots, N_k$ ind. Poisson random measures with average measures $\mu_1, \mu_2, \dots, \mu_k$

$$N = \sum_{i=1}^{k} N_i, \quad N(A) = \sum_{i=1}^{k} N_i(A)$$

$\Rightarrow \forall f_1, \dots, f_k \geq 0$ Borel

$\int f_1 dN_1, \dots, \int f_k dN_k$ are ind.

In particular when $f_1 = \dots = f_k = f$

Theorem → If $N_1, \dots, N_k$ are independent Poisson random measures on $(G, \mathcal{G})$ with average measures $\mu_1, \dots, \mu_k$,

$\Rightarrow N = \sum_{i=1}^{k} N_i$ is also a Poisson random measure with average measure $\mu = \sum_{i=1}^{k} \mu_i$.

Martes 2. feb. 2010.

$$\mathbb{P}\{N_1(A)=n_1, \dots, N_k(A)=n_k \mid N(A)=n\} = \begin{cases} \frac{n!}{n_1! \cdots n_k!} \prod_{i=1}^{k} \left(\frac{\mu_i(A)}{\mu(A)}\right)^{n_i} & n = \sum_{i=1}^{k} n_i \\ 0 & \text{o.w.} \end{cases}$$

where $\mu(A) = \sum_{i=1}^{k} \mu_i(A)$.

* $A_1, \dots, A_k$ disjoint

Let N be a random measure with average measure μ

$N_i(A) = N(A \cap A_i)$

Recall that $N_1, \dots, N_k$ are independent random measures if $\forall f_1, \dots, f_k \geq 0$ Borel functions, $\int f_1 dN_1, \dots, \int f_k dN_k$ are independent r.v.

$$\mathbb{E}\{e^{-\sum_i \theta_i \int f_i dN_i}\} = \mathbb{E}\{e^{-\sum_i \theta_i \int_{A_i} f_i dN}\}$$

$$= \mathbb{E}\{e^{-\int \underbrace{\sum_i \theta_i f_i \mathbb{I}_{A_i}}_{\geq 0 \text{ because } \theta_i \geq 0} dN}\}$$

$$= e^{-\int (1 - e^{-\sum_i \theta_i f_i \mathbb{I}_{A_i}}) d\mu}.$$

$$\hookrightarrow 1 - e^{-\sum_i \theta_i f_i \mathbb{I}_{A_i}} = \begin{cases} 1 - e^{-\theta_i f_i} & \text{on } A_i \\ 0 & \text{on } (\bigcup_{i=1}^{k} A_i)^c \end{cases}$$

$$= \sum_{i=1}^{k} (1 - e^{-\theta_i f_i}) \mathbb{I}_{A_i}$$

$$= e^{-\int \sum_{i=1}^{k} (1 - e^{-\theta_i f_i}) \mathbb{I}_{A_i} d\mu}$$

$$= \prod_{i=1}^{k} e^{-\int (1 - e^{-\theta_i f_i}) \mathbb{I}_{A_i} d\mu}$$

$$= \prod_{i=1}^{k} e^{-\int (1 - e^{-\theta_i f_i}) d\mu_i}.$$

denote $\mu_i(A) = \mathbb{E}(N_i(A))$
$= \mathbb{E}\{N(A \cap A_i)\}$
$= \mu(A \cap A_i)$

$$\Rightarrow \mathbb{E}\{e^{-\sum_{i=1}^{k} \theta_i \int f_i dN_i}\} = \prod_{i=1}^{k} e^{-\int (1 - e^{-\theta_i f_i}) d\mu_i}.$$

This implies that $\int f_i \, dN_i$ are independent random variables where $\mathbb{E}\{e^{-\theta_i \int f_i \, dN_i}\} = e^{-\int (1-e^{-\theta_i f_i}) d\mu_i}$.

In particular, setting $\theta_i = 1$, wee see that

$$\mathbb{E}\{e^{-\int f_i \, dN_i}\} = e^{-\int (1-e^{-f_i}) d\mu_i}, \text{ therefore,}$$

N_i is a Poisson random measure with average measure μ_i.

Therefore, $N_1, \ldots, N_k$ are independent Poisson random measures with average measures $\mu_1, \ldots, \mu_k$

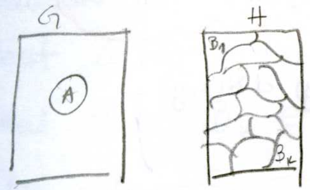
Theorem → Let N be a Poisson random measure with average measure μ. Suppose $A_1, \ldots, A_k$ are disjoint. We define $N_1, \ldots, N_k$ as $N_i(A) = N(A \cap A_i)$,

$\Rightarrow N_1, \ldots, N_k$ are independent Poisson random measures $\mu_1, \ldots, \mu_k$, where $\mu_i(A) = \mu(A \cap A_i)$.

* $(G \times H, \mathcal{G} \otimes \mathcal{H})$.

Take a Poisson random measure N on this space with average measure μ.

Assume that $\tilde{\mu}(A) = \mu(A \times H)$. (a measure on $(G, \mathcal{G})$). is σ-finite.



$\tilde{N}_i(A) = N(A \times B_i)$ is a random measure on $(G, \mathcal{G})$.

$\tilde{\mu}_i(A) = \mu(A \times B_i)$.

$B_1, \ldots, B_k$ partition of H
$\Rightarrow G \times B_1, \ldots, G \times B_k$ partition of $G \times H$

Let $f_1, \ldots, f_k \geq 0$ be Borel functions.

∋. $f_i : G \to \mathbb{R} \ \forall \ i \in \{1, \ldots, k\}$.

We would like to show that, for each choice of such functions, $\{\int f_i \, d\tilde{N}_i\}_{i=1}^k$ are ind r.v. and that $\tilde{N}_i$ are Poisson random measures with average measure $\tilde{\mu}_i$