1

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$(\Omega, \mathcal{F}, P)$

$\mathcal{F}$ sigma-álgebra $\Rightarrow$ 1) $\mathcal{F} \neq \emptyset$

2) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

3) $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{F} \Rightarrow \bigcup_{i\in\mathbb{N}} A_i \in \mathcal{F}$

Corollary

~~Corolario~~ $\rightarrow$ 1) $\mathcal{F}$ closed under any finite or countable sequence of operations of sets $(\cup, \cap, \setminus, \Delta, c)$

symmetric difference:

$A \Delta B = (A\cup B) \cap (A\cap B)^c$

$= (A\cup B) \setminus (A \cap B)$

2) $\emptyset, \Omega \in \mathcal{F}$

$\rightarrow$ $\{\emptyset, \Omega\}$ is the smallest σ-algebra possible

$\rightarrow$ $2^\Omega = \{A \mid A \subseteq \Omega\}$ is the largest ~~greatest~~ σ-algebra possible.

$\rightarrow$ $\{\emptyset, \Omega\} \subseteq \mathcal{F} \subseteq 2^\Omega \quad \forall \quad \mathcal{F}$ σ-algebra.

3) $\mathcal{F}_\theta$ σ-álgebra $\forall \quad \theta \in \Theta$ (not necessarily countable)

$\Rightarrow \bigcap_{\theta\in\Theta} \mathcal{F}_\theta$ σ-algebra

It is the largest σ-field contained in $\mathcal{F}_\theta \ \forall\, \theta \in \Theta$.

4) If G is a family of subsets $(G \subseteq 2^\Omega)$, denote $\sigma(G) = \bigcap_{\mathcal{F}\supseteq G} \mathcal{F} \Rightarrow \sigma(G)$ is the smallest σ-field that contains G.

$\sigma(G) \equiv \sigma$-algebra generated by G.

5) Unions of σ-algebras are not necessarily σ-algebras. Intersections are.

$\rightarrow$ $\mathcal{F}_1 = \{\emptyset, \{1\}, \{2,3\}, \Omega\}$

$\mathcal{F}_2 = \{\emptyset, \{1,2\}, \{3\}, \Omega\}$

Notación $\rightarrow$ $\bigvee_{\theta\in\Theta} \mathcal{F}_\theta = \sigma\left(\bigcup_{\theta\in\Theta} \mathcal{F}_\theta\right)$

P - probabilidad $\rightarrow$ $P: \mathcal{F} \rightarrow [0,1]$.

$\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{F}$ mutually disjoint $\Rightarrow \mathbb{P}\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mathbb{P}(A_i)$

$\mathbb{P}(\Omega) = 1$

Corollary $\rightarrow$ 1) $\mathbb{P}(\emptyset) = 0$.

②

3) $\{A_i\}_{i=1}^{n} \subseteq \mathcal{F} \Rightarrow \mathbb{P}(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} \mathbb{P}(A_i)$.

$\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{F} \Rightarrow \mathbb{P}(\bigcup_{i\in\mathbb{N}} A_i) \leq \sum_{i\in\mathbb{N}} \mathbb{P}(A_i)$.

4) $\{A_i\}_{i=1}^{n} \subseteq \mathcal{F}$ or $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{F}$ $\;\text{s.t.}\;$ $\mathbb{P}(A_i) = 0 \;\forall i$

$\Rightarrow \mathbb{P}(\bigcup_i A_i) = 0$.

5) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

6) $\{A_i\} \subseteq \mathcal{F}$ finite or countable subset

$\mathbb{P}(A_i) = 1 \;\forall i \Rightarrow \mathbb{P}(\bigcap_i A_i) = 1$

7) $A \subseteq B \Rightarrow \mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A)$

$\Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$.

8) $A_1 \subseteq A_2 \subseteq \ldots \in \mathcal{F} \Rightarrow \lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(\bigcup_{n=1}^{\infty} A_n)$

$\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{F} \Rightarrow \lim_{n\to\infty} \mathbb{P}(\bigcup_{i=1}^{n} A_i) = \mathbb{P}(\bigcup_{i\in\mathbb{N}} A_i)$

} Continuity of probability.

$A_1 \supseteq A_2 \supseteq \ldots \in \mathcal{F} \Rightarrow \lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(\bigcap_{n=1}^{\infty} A_n)$

$A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \lim_{n\to\infty} \mathbb{P}(\bigcap_{i=1}^{n} A_i) = \mathbb{P}(\bigcap_{i=1}^{\infty} A_i)$

* Under the assumption that for $A, B \in \mathcal{F}$ disjoint, $\mathbb{P}(A\cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ we have that 8) $\Longleftrightarrow$ σ-aditivity. $\left(\mathbb{P}(\bigcup_{i\in\mathbb{N}} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)\right)$

---

* $\{A_i\}_{i\in\mathbb{N}}$ independent if $\forall n \geq 2$, $i_1 < i_2 < \ldots < i_n$

$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_n}) = \mathbb{P}(A_{i_1}) \cdots \mathbb{P}(A_{i_n})$

Corollary → 1) $\{B_i\}_{i\in\mathbb{N}}$ are ind $\forall$ $B_1, B_2, \ldots$ s.t. $B_i \in \{\emptyset, A_i, A_i^c, \Omega\}$.

$A \perp B$	$A^c \perp B^c$	
$A^c \perp B$	$A \perp \emptyset$	$\emptyset \perp \Omega$.
$A \perp B^c$	$A \perp \Omega$	

2) It is possible that $A_{i_1}, \ldots, A_{i_{n-1}}$ are ind $\forall \{i_1, \ldots, i_{n-1}\} \subseteq \{1, \ldots, n\}$ but $\{A_1, \ldots, A_n\}$ are dependent

3) $\mathbb{P}(\bigcap_{i=1}^{n} A_i) = \prod_{i=1}^{n} \mathbb{P}(A_i) \;\forall n \geq 2 \not\Rightarrow$ independence for the whole independence

$A_1 = \{1, 2, 3, 4\}$

$A_2 = \{3, 4, 5, 6\}$

$A_3 = \{4, 6, 7, 8\}$

Clearly not independent

$\Omega = \{1, \ldots, 8\}$; $\mathbb{P}\{\{i\}\} = 1/8$.

③

Exercise ⊳ $\mathbb{P}(\bigcap_{i=1}^{n} B_i) = \prod_{i=1}^{n} \mathbb{P}(B_i) \quad \forall\ B_i \in \{\emptyset, A_i, A_i^c, \Omega\}$

$\Rightarrow A_1, A_2, \dots$ indep

5) A, B disjoint and independent $\Rightarrow \mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

6) $\mathbb{P}(A) \in \{0, 1\} \rightarrow A \perp B \quad \forall B \in \mathcal{F}$

7) $\{A_i\}_{i \in \mathbb{N}}$ indep. $\quad \mathbb{P}(\bigcap_{i=1}^{\infty} A_i) = \prod_{i=1}^{\infty} \mathbb{P}(A_i)$.

Borel-Cantelli Lemma ⊳ $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{F}$

- $\sum_{i \in \mathbb{N}} \mathbb{P}(A_i) < \infty \Rightarrow \mathbb{P}(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m) = 0.$ $\quad \{w \mid \#\{n \mid w \in A_n\} = \infty\}$

  $\underbrace{}_{} A_n$ i.o. ← infinitely often

- $\sum_{i \in \mathbb{N}} \mathbb{P}(A_i) = 1 \Rightarrow \mathbb{P}(A_n \text{ i.o.}) = 1.$

Borel sets of $\mathbb{R}$ ⊳ smallest σ-algebra that contains $(a, b)\ \forall\ a < b \in \mathbb{R}$.

Product σ-algebra ⊳ $\mathcal{F}_1 \otimes \mathcal{F}_2 = \sigma(\{A \times B \mid A \in \mathcal{F}_1, B \in \mathcal{F}_2\})$

★ Measure

$(\Omega, \mathcal{F})$ measurable space

* $\mu : \mathcal{F} \rightarrow \mathbb{R}^+ \cup \{0, \infty\}$.

$\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{F}$ disjoint $\Rightarrow \mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$

$\mu(\emptyset) = 0.$

* A measure is called σ-finite if

1) $\exists \{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{F}$ mutualy disjoint $\therefore \bigcup_{i \in \mathbb{N}} A_i = \Omega$

$\therefore \mu(A_i) < \infty \quad \forall i \in \mathbb{N}$.

¿or?

2) $\exists A_1 \subseteq A_2 \subseteq \dots \in \mathcal{F} \quad \bigcup_{i \in \mathbb{N}} A_i = \Omega \therefore \mu(A_i) < \infty \quad \forall i \in \mathbb{N}$

Remark ⊳ μ σ-finite, $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{F}$ mutually disjoint with $\bigcup_{i=1}^{\infty} A_i = \Omega$, $\mu(A_i) < \infty$

* $\lambda_i = \mu(A_i)$, $\mu(A) = \sum_{i=1}^{\infty} \lambda_i \mathbb{P}_i(A_i)$

→ Any σ-finite measure is a non-negative linear combination of probability measures.

↳ $0 < \mu(\Omega) < \infty \Rightarrow \mathbb{P}(A) = \frac{\mu(A)}{\mu(\Omega)} \quad \lambda = \mu(\Omega) \Rightarrow \mu(A) = \lambda \mathbb{P}(A)$.

④

Lebesgue measure → it's a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$
↑ Borel of $\mathbb{R}$

that satisfies $\ell((a,b)) = b-a$.

* $\ell(A) = 0 \iff \forall \varepsilon>0\ \exists\ \{a_i, b_i\}_{i\in\mathbb{N}}$, $a_i < b_i$ ∋ $\bigcup_{i\in\mathbb{N}} (a_i, b_i) \supset A$, $\sum_{i=1}^{\infty} (b_i - a_i) < \varepsilon$.

* ℓ is σ-finite ← $\ell((n-1, n]) = 1$, $\bigcup_{n\in\mathbb{Z}} (n-1, n] = \mathbb{R}$.

Product measures → $(\Omega_1, \mathcal{F}_1, \mu_1)$, $(\Omega_2, \mathcal{F}_2, \mu_2)$ σ-finite measure spaces.

$\mu_1 \otimes \mu_2$ is called the product measure and it is the unique measure on $\mathcal{F}_1 \otimes \mathcal{F}_2$ ∋ $\mu_1\otimes\mu_2(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$ $\forall A_1 \in \mathcal{F}_1,\ A_2 \in \mathcal{F}_2$.

* $\mathbb{E}(X) = \mathbb{E}\left(\int_0^X dt\right) = \mathbb{E}\left(\int_{\mathbb{R}^+} \mathbb{I}_{[0,x)}(t)\, dt\right) \overset{?}{=} \int_{\mathbb{R}^+} \mathbb{E}\left(\mathbb{I}_{[0,x)}(t)\right) dt$
$X > 0$
↑ $\mathbb{I}_{\{x>t\}}$
$= \int_0^{\infty} \mathbb{P}(X > t)\, dt$

Martes 26 enero

Notation → $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$

$X: \Omega \to \overline{\mathbb{R}}$ ← (extended) random variable.

↳ X is a random variable if $\forall\ t \in \mathbb{R}$
$\{X \le t\} = \{w : X(w) \le t\} = X^{-1}([-\infty, t]) \in \mathcal{F}$
↑ we want measurable events.

Theorem → X is a r.v.
⇒ $\forall\ B$ Borel subset of $\mathbb{R}$, we have that
$\{X \in B\} = \{w : X(w) \in B\} = X^{-1}(B) \in \mathcal{F}$

$X^{-1}(B^c) = (X^{-1}(B))^c$
$X^{-1}(\bigcup_\theta B_\theta) = \bigcup_\theta X^{-1}(B_\theta)$
$X^{-1}(\bigcap_\theta B_\theta) = \bigcap_\theta X^{-1}(B_\theta)$

* $(\Omega_1, \mathcal{F}_1) \xrightarrow{X} (\Omega_2, \mathcal{F}_2) \xrightarrow{Y} (\Omega_3, \mathcal{F}_3)$
$X^{-1}(A_2) \in \mathcal{F}_1\ \forall\ A_2 \in \mathcal{F}_2$ ⇒ X is a measurable transformation.
So is Y if $Y^{-1}(A_3) \in \mathcal{F}_2\ \forall\ A_3 \in \mathcal{F}_3$.

⑤.

* $f: \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ is a Borel function if

$$(\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}})) \xrightarrow{f} (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}})) \quad \text{measurable.}$$

* A random variable is a measurable transformation $(\Omega, \mathcal{F}) \xrightarrow{X} (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$.

Corollary ▷ If $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of r.v.

$\Rightarrow$ so are $\sup\{X_n\}_{n\in\mathbb{N}}$, $\inf\{X_n\}_{n\in\mathbb{N}}$, $\overline{\lim_{n\to\infty}} X_n$, $\underline{\lim_{n\to\infty}} X_n$

and, if the limit $\exists$, then so is $\lim_{n\to\infty} X_n$.

$$\overline{\lim_{n\to\infty}} X_n = \inf_{n\geq 1} \sup_{m \geq n} X_m$$

▷ f Borel, X r.v. $\Rightarrow$ $f(X)$ r.v.

More generally, if $f: (\bar{\mathbb{R}}^n, \mathcal{B}(\bar{\mathbb{R}}^n)) \longrightarrow (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$ Borel

$X_1, \dots, X_n$ r.v.

$\Rightarrow f(X_1, \dots, X_n)$ r.v.

$$\mathcal{B}(\bar{\mathbb{R}}^n) = \mathcal{B}(\bar{\mathbb{R}}) \otimes \cdots \otimes \mathcal{B}(\bar{\mathbb{R}})$$ not by definition, it's a theorem.

* $\mathbb{E}(I_A) = \mathbb{P}(A)$

$$\mathbb{E}\Big(\underbrace{\sum_{i=1}^n a_i I_{A_i}}\Big) = \sum_{i=1}^n a_i \, \mathbb{P}(A_i)$$

Simple (measurable) function

X	P
a_1	$\mathbb{P}(A_1)$
a_2	$\mathbb{P}(A_2)$
$\vdots$	$\vdots$
a_n	$\mathbb{P}(A_n)$
0	$1-\sum_{i=1}^{n}\mathbb{P}(A_i)$

* $0 \leq X$, $\quad X_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \mathbb{I}_{\{\frac{i-1}{2^n} < X < \frac{i}{2^n}\}} \leq X$

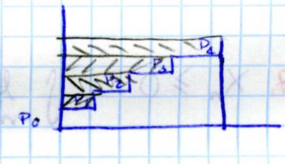
On $\{X \leq n2^n\}$, $0 \leq X - X_n \leq \frac{1}{2^n} \xrightarrow[n\to\infty]{} 0$

$X_n \leq X_{n+1}$

$X_n \to X \qquad \mathbb{E}(X_n) = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \mathbb{P}\left(\frac{i-1}{2^n} < X \leq \frac{i}{2^n}\right)$

* $X \geq 0 \Rightarrow \mathbb{E}(X) = \lim_{n\to\infty} \mathbb{E}(X_n)$

$$\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > t)\, dt = \int_0^\infty (1 - F_X(t))\, dt.$$



$$= \int_{\bar{\mathbb{R}}^+} X(\omega)\, dP(\omega) = \int_{\bar{\mathbb{R}}^+} x \, dF_X(x) = \int x \, d\mathbb{P}_X(x)$$

$\mathbb{P}_X(B) = \mathbb{P}(X \in B) = \mathbb{P}(X^{-1}(B))$

⑥. * $X \le 0$.

$$\mathbb{E}(X) = -\mathbb{E}(-X) = -\int_{\mathbb{R}^+} \mathbb{P}(-X > t)\,dt = -\int_{\mathbb{R}^-} \mathbb{P}(X < t]\,dt = -\int_{\mathbb{R}^-} \mathbb{P}(X < t)\,dt.$$

$$= -\int_{[\infty,0]} F_X(t)\,dt.$$

* If $\mathbb{E}(X^+) = \int X\, \mathbb{I}_{\{X \ge 0\}}\,dP = \mathbb{E}(X \mathbb{I}_{\{X \ge 0\}}) = \int_{\mathbb{R}^+} (1 - F_X(x))\,dx = \int_{[0,\infty]} x\,dF_X(x) < \infty$

or. $\mathbb{E}(X^-) = \int X\, \mathbb{I}_{\{X < 0\}}\,dP\} = \mathbb{E}(X \mathbb{I}_{\{X \le 0\}}) = -\int_{[\infty,0]} F_X(x)\,dx = \int_{[\infty,0]} x\,dF_X(x) > -\infty$.

$\Rightarrow$ we say that $\mathbb{E}(X)\ \exists$ (it could be ∞ or $-\infty$).

* $\mathbb{E}(X) = \int_{\mathbb{R}} x\,dF_X(x) = \int X\,dP = -\int_{-\infty}^{0} F_X(x)\,dx + \int_0^{\infty} (1 - F_X(x))\,dx$.

* If μ σ-finite, $\mu(A) = \sum_{i=1}^{\infty} \lambda_i \mathbb{P}_i(A)$

$\mathbb{E}_i(X) = \int X\,dP_i$

if $X \ge 0 \Rightarrow \int X\,d\mu = \sum_{i=1}^{\infty} \lambda_i \mathbb{E}_i(X)$

* if $\int X^+ d\mu < \infty$ or $\int (X^-)\,d\mu < \infty$

$\Rightarrow \int X\,d\mu = \int X^+ d\mu - \int (-X^-)\,d\mu$

* $\mu_X(A) = \mu(X^{-1}(A))$

Properties of integration.

1. $\int_A d\mu = \int \mathbb{I}_A\,d\mu = \mu(A)$
2. $\int aX\,d\mu = a\int X\,d\mu$
3. $\int (X+Y)\,d\mu = \int X\,d\mu + \int Y\,d\mu$, provided that not both are infinite with different signs.
4. $X \le Y \Rightarrow \int X\,d\mu \le \int Y\,d\mu$.
5. $0 \le X_n \uparrow X \Rightarrow \int X_n\,d\mu \xrightarrow[n\to\infty]{} \int X\,d\mu$

   $\boxed{\text{Monotone convergence theorem}}$

5'. $X_n \ge 0 \Rightarrow \int \sum_{n=1}^{\infty} X_n\,d\mu = \sum_{n=1}^{\infty} \int X_n\,d\mu$.

6. $|X_n| \le Y$, $\int Y\,d\mu < \infty$.

   $\mu\{X_n \not\xrightarrow[n\to\infty]{} X\} = 0$. $\Rightarrow \int X_n\,d\mu \xrightarrow[n\to\infty]{} \int X\,d\mu$

   $\boxed{\text{Dominated convergence theorem}}$

7. $X_n \ge 0$ $\int \varliminf_{n\to\infty} X_n\,d\mu \le \varliminf_{n\to\infty} X_n\,d\mu$.

   $\boxed{\text{Fatou's Lemma}}$

(7).

Theorem ▷ **Fubini / Tonelli** ▷ $f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$(\Omega_i, \mathcal{F}_i, \mu_i)$; $i \in \{1, 2\}$.

$$f \geq 0 \Rightarrow \int f \, d\mu_1 \otimes \mu_2 = \int_{\Omega_2} \int_{\Omega_1} f(x, y) \, d\mu_1(x) \, d\mu_2(y)$$

$$= \int_{\Omega_1} \int_{\Omega_2} f(x, y) \, d\mu_2(y) \, d\mu_1(x)$$

↳ $\mathbb{E}(X) = \int_{\Omega} \int_{\mathbb{R}} \mathbb{I}_{\{X(\omega) > t\}} \, dt \, dP(\omega)$

$$= \int_{\mathbb{R}} \int_{\Omega} \mathbb{I}_{\{X(\omega) > t\}} \, dP(\omega) \, dt = \int_0^{\infty} \mathbb{P}(X > x) \, dx.$$

↳ We just have to see if $\mathbb{I}_{\{X(\omega) > t\}}$ is jointly measurable.

$$\{(\omega, t) \mid X(\omega) > t\} = \bigcup_{q \in \mathbb{Q}} \underbrace{\{X(\omega) > q\}}_{\in \mathcal{F}} \times \underbrace{\{q > t\}}_{\in \mathcal{B}(\mathbb{R})} \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R})$$

* For general f, if either $\int f^+ \, d\mu_1 \otimes d\mu_2 < \infty$ or $\int (-f^-) \, d\mu_1 \otimes d\mu_2 < \infty$.
  ⇒ the same holds.
  * $\int_{\Omega_2} f(x, y) \, d\mu_2(y)$ $\mathcal{F}_1$-measurable function of x

    $\int_{\Omega_1} f(x, y) \, d\mu_1(x)$ $\mathcal{F}_2$-measurable function of y.

Conditional expectation.

$(\Omega, \mathcal{F}, \mathbb{P})$ $\quad \mathcal{F}_0 \subseteq \mathcal{F}$

$Y = \mathbb{E}(X \mid \mathcal{F}_0)$ $\quad$ Y is $\mathcal{F}_0$ measurable.

$Y \in \mathcal{F}_0$

$\{Y \leq t\} \in \mathcal{F}_0 \ \forall t$.

$$\forall A \in \mathcal{F}_0, \ \mathbb{E}(Y \mathbb{I}_A) = \mathbb{E}(X \mathbb{I}_A)$$

Theorem ▷ $\mathbb{E}(X \mid \mathcal{F}_0)$ is well defined and is determined with probability 1. provided that $\mathbb{E}(X) \ \exists$, $(\mathbb{E}(X^+) < \infty$ or $\mathbb{E}(X^-) > -\infty)$

* The conditional expected value has the same properties as ordinary expected value.

Def ▷ $\sigma(X)$: smallest σ-field for which X is measurable.
  * $\{X^{-1}(B) \mid B \in \mathcal{B}(\mathbb{R})\}$ is already a σ-algebra.

* for every r.v. Z measurable respect to $\sigma(X)$, $\exists f$ Borel $\ni$: $Z = f(X)$.
  ↳ $\mathbb{E}(Y \mid X) = \mathbb{E}(Y \mid \sigma(X)) = f(X)$.

⑧ **Another form of Borel-Cantelli.**

$\Rightarrow \mathbb{E}\{\sum_{n=1}^{\infty} \mathbb{I}_{A_n}\} < \infty \Rightarrow \mathbb{P}\{\sum_{n=1}^{\infty} \mathbb{I}_{A_n} < \infty\} = 1$

$= \infty \Rightarrow \mathbb{P}\{\text{idem}\} = 0$, $A_1, A_2, \dots$ ind.

* 1) $\{X_n\}_{n\in\mathbb{N}}$ independent if $\forall k \geq 2$, $\forall\ t_1, \dots, t_k$,

$\mathbb{P}\{X_1 \leq t_1, \dots, X_k \leq t_k\} = \prod_{i=1}^{k} \mathbb{P}\{X_k \leq t_k\}$.

2) $\forall\ k \in \mathbb{N}$, $B_1, \dots, B_k \in \mathcal{B}(\mathbb{R})$

$\mathbb{P}(X_1 \in B_1, \dots, X_k \in B_k) = \prod_{i=1}^{k} \mathbb{P}(X_i \in B_i)$.

3) $\forall k$, Borel-functions $f_1, \dots, f_k$.

$\mathbb{E}(f_1(X_1) \cdots f_k(X_k)) = \prod_{i=1}^{k} \mathbb{E}(f_i(X_i))$

4) $\mathbb{E}\{e^{i(\sum_{j=1}^{k} \theta_j X_j)}\} = \prod_{j=1}^{k} \mathbb{E}\{e^{i\theta_j X_j}\}$ $\forall\ k, \theta_1, \dots, \theta_k \in \mathbb{R}$.

5) $X_k \geq 0$ $k = 1, 2, \dots$ $\mathbb{E}\{e^{-\sum_{k=1}^{n} \theta_k X_k}\} = \prod_{k=1}^{n} \mathbb{E}\{e^{-\theta_k X_k}\}$.

——— o ——— (End of review).

$N \sim \text{Poisson}(\lambda) \Rightarrow \mathbb{P}(N = n) = \frac{e^{-\lambda}\lambda^n}{n!}$

$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \sim \text{Multinomial}(n, p_1, \dots, p_k) \Rightarrow \mathbb{P}(X_1 = n_1, \dots, X_k = n_k) = \frac{(n_1 + \dots + n_k)!}{n_1! \cdots n_k!} p_1^{n_1} \cdots p_k^{n_k}$

* Assume $N \sim \text{Poisson}(\lambda)$

$\begin{pmatrix} Y_1 \\ \vdots \\ Y_k \end{pmatrix} \Big| N = n \sim \text{Multinomial}(n, p_1, \dots, p_k)$

Question → find the joint distribution of $\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$ (unconditional.

$\mathbb{P}(Y_1 = n_1, \dots, Y_k = n_k) = \sum_{n\in\mathbb{N}\cup\{0\}} \underbrace{\mathbb{P}\{Y_1 = n_1, \dots, Y_k = n_k, N = n\}}_{= 0 \text{ if } n \neq n_1 + \dots + n_k}$

$= \sum_{n\in\mathbb{N}\cup\{0\}} \mathbb{P}\{Y_1 = n_1, \dots, Y_k = n_k \mid N = n\} \mathbb{P}\{N = n\}$.

$= \mathbb{P}\{Y_1 = n_1, \dots, Y_k = n_k \mid N = \sum_{i=1}^{k} n_i\} \mathbb{P}\{N = \sum_{i=1}^{k} n_i\}$.

$= \left\{\frac{\cancel{(n_1 + \dots + n_k)!}}{n_1! \cdots n_k!} p_1^{n_1} \cdots p_k^{n_k}\right\} \frac{e^{-\lambda} \lambda^{\sum_{i=1}^{k} n_i}}{\cancel{(\sum n_i)!}}$ $\qquad \lambda = \sum_{i=1}^{k} p_i \lambda$, $\quad \lambda^{\sum_{i=1}^{k} n_i} = \prod_{i=1}^{k} \lambda^{n_i}$

$= \prod_{i=1}^{k} \frac{e^{-\lambda p_i} (\lambda p_i)^{n_i}}{n_i!}$ ← producto de k Poisson (λp_i) indep.

$\Rightarrow Y_i \sim \text{Poisson}(\lambda p_i)$; $Y_1, \dots, Y_k$ indep.

(9) Miércoles 27 enero 2010.

* $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$, $X \perp Y$

$\Rightarrow X + Y \sim \text{Poisson}(\lambda+\mu)$

* By induction, if $\{N_i\}_{i=1}^{n} \sim \text{Poisson}(\lambda_i)$ ind

$\Rightarrow \sum_{i=1}^{n} N_i \sim \text{Poisson}(\sum_{i=1}^{n} \lambda_i)$

* Under the same contition, $N_1, \dots, N_k \mid \sum_{i=1}^{k} N_i = n \sim ?$

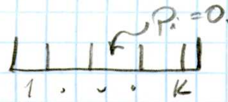
$$\mathbb{P}\{N_1 = n_1, \dots, N_k = n \mid \sum_{i=1}^{k} N_i = n\} = \begin{cases} 0 & n \neq n_1 + \dots + n_k \\ \dfrac{\mathbb{P}(N_1 = n_1, \dots, N_k = n_k)}{\mathbb{P}(\sum_{i=1}^{} N_i = \sum_{i=1}^{} n_i)} & n = n_1 + \dots + n_k \end{cases}$$

$$\frac{\mathbb{P}(N_1 = n_1, \dots, N_k = n_k)}{\mathbb{P}(\sum_{i=1}^{k} N_i = \sum_{i=1}^{k} n_i)} = \frac{\prod_{i=1}^{k} \cancel{e^{-\lambda_i}} \lambda_i^{n_i}/n_i!}{\cancel{e^{-\sum_{i=1}^{k} \lambda_i}} \dfrac{(\sum_{i=1}^{k} \lambda_i)^{\sum n_i}}{(\sum_{i=1}^{} n_i)!}}$$

$$= \frac{(n_1 + \dots + n_k)!}{n_1! \cdots n_k!} \prod_{i=1}^{n} \left(\frac{\lambda_i}{\lambda_1 + \dots + \lambda_k} \right)^{n_i}$$

$\therefore N_1, \dots, N_k \mid \sum_{i=1}^{k} N_i = n \sim \text{Multinomial}(n, p_1, \dots, p_k)$; $p_i = \frac{\lambda_i}{\lambda_1 + \dots + \lambda_k}$

* $Y \sim \text{Poisson}(0) \leftarrow$ what will happen?



$N \sim \text{Poisson}(\lambda)$; $\lambda \in \mathbb{R}^+$

Choose **N** balls and thraw them in the K urns.

$\therefore Y \sim \text{Poisson}(0) \iff \mathbb{P}(Y=0) = 1$.

* $N \sim \text{Poisson}(\lambda) \Rightarrow \mathbb{E}(N) = \text{Var}(N) = \lambda$

$$\mathbb{E}(z^N) = \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} z^n = e^{-\lambda} \underbrace{\sum_{n=0}^{\infty} \frac{(\lambda z)^n}{n!}}_{e^{\lambda z}} = e^{-\lambda(1-z)}$$

* $N_\lambda \sim \text{Poisson}(\lambda)$

$\Rightarrow \lim_{\lambda \to \infty} \mathbb{P}\{N_\lambda \leq n\} = \lim_{n \to \infty} \sum_{i=0}^{n} e^{-\lambda} \frac{\lambda^i}{i!} = 0$

$\Rightarrow \lim_{\lambda \to \infty} \mathbb{P}\{N_\lambda > n\} = 1$

* $\sum_{i=1}^{\infty} \lambda_i = \infty$, $N_i \sim \text{Poisson}(\lambda_i)$

$\Rightarrow \sum_{i=1}^{\infty} N_i \sim \text{Poisson}(\sum_{i=1}^{\infty} \lambda_i)$, $\sum_{i=1}^{k} N_i \uparrow \sum_{i=1}^{\infty} N_i$

$\mathbb{P}\{\sum_{i=1}^{\infty} N_i > n\} = \lim_{n \to \infty} \mathbb{P}\{\sum_{i=1}^{k} N_i > n\}$ $= 1$

$A_k = \{\sum_{i=1}^{k} N_i > n\}$; $A_k \subseteq A_{k+1}$

$\bigcup A_k = \{\sum_{i=1}^{\infty} N_i > n\} \Rightarrow \lim_{k \to \infty} \mathbb{P}(A_k) = \mathbb{P}(\bigcup A_k)$

⑩ * $\mathbb{P}\{\sum_{i=1}^{\infty} N_i > n\} = 1 \quad \forall n$

$\Rightarrow \mathbb{P}\left(\bigcap_{n=0}^{\infty}\{\sum_{i=1}^{\infty} N_i > n\}\right) = 1.$

$= \mathbb{P}(\sum_{i=1}^{\infty} N_i = \infty)$

$\therefore N_1, N_2, \ldots$ ind ; $N_i \sim \text{Poisson}(\lambda_i)$, $\sum_{i=1}^{\infty} \lambda_i = \infty$

$\Rightarrow \mathbb{P}(\sum_{i=1}^{\infty} N_i = \infty) = 1$

* For this reason, we define $N \sim \text{Poisson}(\infty)$ if $\mathbb{P}(N=\infty) = 1$.

* If $\sum_{i=1}^{\infty} \lambda_i < \infty$

$\Rightarrow \mathbb{P}\{\sum_{i=1}^{k} N_i = n\} = e^{-\sum_{i=1}^{k}\lambda_i} \frac{(\sum_{i=1}^{k}\lambda_i)^n}{n!} \xrightarrow[k\to\infty]{} e^{-\sum_{i=1}^{\infty}\lambda_i} \frac{(\sum_{i=1}^{\infty}\lambda_i)^n}{n!}$

Conclusion

$_n \sim \text{Poisson}(\lambda_n)$; $\lambda_n \xrightarrow[n\to\infty]{} \lambda$

$\Rightarrow Y_n \xrightarrow{d} Y \sim \text{Poisson}(\lambda)$

Moreover, if $N_1, N_2, \ldots$ iid con $N_i \sim \text{Poisson}(\lambda_i)$

$\Rightarrow \sum_{i=1}^{\infty} N_i \sim \text{Poisson}(\sum_{i=1}^{\infty}\lambda_i)$

---

* 

$\sum_{i=1}^{\infty} p_i = 1$

∞ urns.

Choose N balls, where $N \sim \text{Poisson}(\lambda)$, $\lambda \in \mathbb{R}^+$
Throw them independently in the urns, where the probability of falling in urn i is p_i.

$\Rightarrow \mathbb{P}(Y_1 = n_1, Y_2 = n_2, \ldots \mid N = n) = \frac{n!}{\prod_{i=1}^{\infty} n_i} \prod_{i=1}^{\infty} p_i^{n_i}$

For this purpose, we define $0^0 = 1$.

$\Rightarrow Y_1, Y_2, \ldots$ ind r.v. with $Y_i \sim \text{Poisson}(\lambda p_i)$

$Y_1, \ldots, Y_k$ ind $\forall k \geq 2$?

$Y_1, \ldots, Y_k, \sum_{i=k+1}^{\infty} Y_i \mid N = n \sim \text{Multinomial}(n, p_1, \ldots, p_k, \sum_{i=k+1}^{\infty} p_i)$

$\sim \text{Multinomial}(n, p_1, \ldots, p_k, 1 - \sum_{i=1}^{k} p_i)$

$k+1$ urns.

$\Rightarrow Y_1, \ldots, Y_k, \sum_{i=k+1}^{\infty} Y_i$ ind, $Y_i \sim \text{Poisson}(\lambda p_i)$