Problem Set # 1

(1) For the simple regression model:

$$y_i = \mu + \varepsilon_i, \text{ with } \varepsilon_i \sim \mathcal{N}(0, \sigma^2),$$

prove that the sample mean is consistent and asymptotically normal.
Now, consider the alternative estimator:

$$\widetilde{\mu} = \sum_{i=1}^{n} w_i y_i, \text{ where } w_i = \frac{i}{\sum_{i=1}^{n} i}.$$

Prove that this is a consistent estimator of μ and obtain its asymptotic variance.

(2) For the regression model:

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \text{ with } \boldsymbol{\varepsilon} \sim \mathcal{N}(\boldsymbol{0}, \sigma^2 \boldsymbol{I}),$$

(a) Obtain the mean and the variance of the ML estimator, $\widehat{\sigma}^2 = \widehat{\boldsymbol{\varepsilon}}' \widehat{\boldsymbol{\varepsilon}}/n$, of σ^2.
(b) Prove that this estimator is consistent.

(3) Consider the following regression model:

$$y_i = \beta_0 + \beta_1 i^\gamma + \varepsilon_i,$$

where the $i = 1, 2, \ldots, n$ and the $\{\varepsilon_i\}$ are such that $\mathrm{E}[\varepsilon_i] = 0$, $\mathrm{E}[\varepsilon_i \varepsilon_j] = \sigma^2 I(i = j)$ and $\sigma > 0$. Prove that the OLS estimator of (β_0, β_1) converges in quadratic mean if and only if $-1/2 < \gamma < 0$ or $\gamma > 0$.

(4) Let be $\{X_i\}$ and $\{Y_i\}$ sequences of i.i.d. random variables, such that:

$$\mathrm{E}[X_i] = 0, \ \mathrm{Var}(X_i) = 1, \ \mathrm{E}[Y_i] = 6 \text{ and } \mathrm{Var}(Y_i) = 5.$$

Let $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$ and $\bar{Y} = n^{-1} \sum_{i=1}^{n} Y_i$. Find the probability orders of:
(a) $\bar{X}$.
(b) $\bar{Y}$.
(c) $\bar{X}\bar{Y}$.
(d) $\bar{X} + 2$.

(5) Write a MATLAB function that illustrates the asymptotic normality of the OLS estimator under i.i.d. non-gaussian innovations.