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A directional test of exponentiality based on maximum correlations

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Abstract In Fortiana and Grané (J Stat Plann Infer 108:85–97), we study a scale-free statistic, based on Hoeffding's maximum correlation, for testing exponentiality. This statistic admits an expansion along a countable set of orthogonal axes, originating a sequence of statistics. Linear combinations of a given number p of terms in this sequence can be written as a quotient of L-statistics. In this paper, we propose a scale-free adaptive statistic for testing exponentiality with optimal power against a specific alternative and obtain its exact distribution. An empirical power study shows that the test based on this new statistic has the same level of performance than the best tests in the statistical literature.

Keywords Goodness-of-fit · Exponentiality · Maximum correlation · Decomposition of tests statistics · Exact distribution · L-statistics

1 Introduction

The exponential distribution is probably the one most used in statistical work after the normal distribution. It often appears in problems dealing with life testing, reliability

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theory and the theory of stochastic processes. In those contexts it is usually assumed that the data involved in the experiment are exponentiality distributed. Stephens and D'Agostino (1986) and Doksum and Yandell (1984) contain a detailed review of tests of exponentiality.

The family of tests we are concerned with in this paper is based on Hoeffding's maximum correlation between two probability distribution functions F_1 and F_2, with second order moments, which is defined as the correlation coefficient of the *Fréchet upper bound* of F_1 and F_2, that is:

$$\rho^+(F_1, F_2) = \frac{1}{\sigma_1\,\sigma_2}\left(\int_0^1 F_1^-(p)\,F_2^-(p)\,dp - \mu_1\,\mu_2\right), \tag{1}$$

where F_i^- is the left-continuous pseudoinverse of F_i, $\mu_i = E(F_i)$ and $\sigma_i^2 = var(F_i)$, $i = 1, 2$. (see, e.g., Cambanis et al. 1976). Here the notation $E(F)$ represents the expected value of any random variable whose probability distribution function is F, and analogously for $var(F)$. Since $\rho^+(F_1, F_2)$ equals 1 if and only if $F_1 = F_2$ (almost everywhere) up to a scale and location change, the quantity $\rho^+(F_n, F_0)$ has been used in previous works, Fortiana and Grané (2002, 2003); Grané and Fortiana (2006, 2008) and Grané and Fortiana (2009) as a qualitative measure of goodness-of-fit of an iid sample, with empirical distribution function F_n, to a given distribution F_0. More precisely, in Fortiana and Grané (2002) we defined and studied the properties of

$$Q_n = \frac{s_n}{\bar{y}_n}\,\rho^+(F_n, F_0), \tag{2}$$

where $\bar{y}_n$ and s_n^2 are, respectively, the sample mean and variance, as a goodness-of-fit statistic for testing the composite hypothesis of exponentiality when the scale parameter is unknown, that is

$$H_0 : F = F_0 \equiv Exp(0, \beta), \quad \beta \in \mathbb{R}_+, \tag{3}$$

where $Exp(0, \beta)$ is the exponential distribution with location parameter $\alpha = 0$ and unknown scale parameter $\beta > 0$, $F_0(x) = 1 - \exp(-x/\beta)$, $x \geq 0$. We found that Q_n has reasonably good properties as a goodness-of-fit test and we also computed its exact distribution under the null hypothesis. A recent work of Tchirina (2007) studies the local asymptotic optimality of the test based on Q_n.

In the present work we prove the following identity

$$Q_n = \sum_{j\geq 0} \omega_j\,\tilde{\Phi}_{nj}$$

where the sequence $\{\tilde{\Phi}_{nj}\}_{j\geq 0}$ appears as a decomposition of this test, analogous to those studied by Durbin and Knott (1972, 1975) and Stephens (1974). The statistics $\tilde{\Phi}_{nj}$'s are scale-free and they are obtained as a slight modification of the corresponding

jth Fourier coefficient of the pseudoinverse of F_n for the orthonormal (in $L^2[0,1]$) sequence $\{\phi_j(t)\}_{j\geq 0}$, $t \in [0,1]$ (see Propositions 2.1 and 2.2).

In this article, we seek to improve the performance of Q_n for a specific alternative by choosing a proper linear combination of a finite number p of terms of the scale-free sequence of statistics $\{\tilde{\Phi}_{nj}\}_{j\geq 0}$. We call this linear combination the scale-free adaptive statistic and we denote it by T_p.

Section 3 contains the construction of the T_p statistic for testing (3) against a specific alternative. Power optimization is translated into an eigenvalue-type problem with quadratic forms, functions of the first two moments of the order statistic. Since it is possible that for certain families of alternatives some of the required second order moments do not exist, in Sect. 4 we try to circumvent this problem using the asymptotic theory of L-statistics in order to obtain approximations of the quadratic forms involved in Sect. 3. In Sect. 5 we obtain the exact distribution of T_p and, as an illustration, in Sect. 6 we perform the actual computations for the orthonormal basis of the Legendre polynomials, comparing the power of T_p with other scale-free statistics for testing exponentiality such as the Stephens' modification of the Shapiro–Wilk statistic Stephens (1978), the Gini statistic of Gail and Gastwirth (1978), the Cox and Oakes (1984) statistic, the statistic of Epps and Pulley (1986), the statistic proposed by Ebrahimi et al. (1992) based on the Vasicek's entropy estimator, the statistic of Henze (1993), the Kolmogorov–Smirnov and Cramér–von Mises type statistics proposed by Baringhaus and Henze (2000), the statistics of Henze and Meintanis (2002) based on the empirical characteristic function, smooth tests for testing exponentiality (Rayner and Best 1989) and the Q_n statistic of Fortiana and Grané (2002). In Sect. 7 we investigate the performance of the T_p statistic, designed to distinguish a particular alternative, in detecting other departures from the exponentiality. Although the test based on T_p is not adaptive in the usual sense, it can be regarded as some kind of compromise between an omnibus test and a directional test. Through an empirical power study we have seen that the test based on T_p has the same level of the performance as the best tests of exponentiality in the statistical literature.

2 Definition of the statistic

Consider n iid random variables with probability distribution function F, with finite second order moment, and empirical distribution function F_n. Let μ and σ^2 be the expectation and variance of F, respectively. Let $\{\phi_j(t)\}_{j\geq 0}$ be an orthonormal sequence in $L^2[0,1]$, $Exp(0,\beta)$ the exponential distribution with scale parameter $\beta > 0$ and location parameter $\alpha = 0$.

Proposition 2.1 *The Hoeffding maximum correlation has the following decomposition:*

$$\rho^+(F, Exp(0,\beta)) = \frac{1}{\sigma}\sum_{j\geq 0}\omega_j\,\Phi_j(F),$$

where $\Phi_j(F) = \int_0^1 F^-(t)\,\phi_j(t)\,dt$ *and* $\omega_j = \int_0^1 -(1+\log(1-t))\,\phi_j(t)\,dt$, *for* $j \geq 0$.

Proof Noticing that $\mu = \int_0^1 F^-(t)\,dt$, formula (1) can be written as

$$\rho^+(F, Exp(0,\beta)) = \frac{1}{\sigma\,\beta}\left(\int_0^1 -F^-(t)\,\beta\log(1-t)\,dt - \mu\,\beta\right)$$
$$= \frac{1}{\sigma}\int_0^1 -(1+\log(1-t))\,F^-(t)\,dt$$
$$= \frac{1}{\sigma}\sum_{j\geq 0}\omega_j\int_0^1 F^-(t)\,\phi_j(t)\,dt,$$

where we have expanded function $-(1+\log(1-t))$ in a Fourier series with respect to the orthonormal sequence $\{\phi_j(t)\}_{j\geq 0}$. □

Let $y_{(1)},\ldots,y_{(n)}$ be the order statistic obtained from n iid$\sim$ $Exp(0,\beta)$ random variables with empirical distribution function F_n. For $j \geq 0$ we define the sequence of L-statistics

$$\Phi_{nj} \equiv \Phi_j(F_n) = \int_0^1 F_n^-(t)\,\phi_j(t)dt = \frac{1}{n}\sum_{i=1}^n a_{ij}\,y_{(i)},$$

where $a_{ij} = n\int_{(i-1)/n}^{i/n}\phi_j(t)\,dt$, and the sequence of scale-free statistics is defined as $\tilde{\Phi}_{nj} = \Phi_{nj}/\bar{y}_n$.

Proposition 2.2 *With the previous notation, the statistic Q_n of (2) is scale-free and can be written as $Q_n = \sum_{j\geq 0}\omega_j\,\tilde{\Phi}_{nj}$.*

Proof Note that Q_n is written as a sequence of scale-free statistics, since each $\tilde{\Phi}_{nj}$ is a quotient of L-statistics. From Proposition 2.1 and using F_n instead of F, we have that

$$Q_n = \frac{s_n\,\rho^+(F_n, Exp(0,\beta))}{\bar{y}_n} = \frac{\sum_{j\geq 0}\omega_j\,\Phi_{nj}}{\bar{y}_n} = \sum_{j\geq 0}\omega_j\,\tilde{\Phi}_{nj}.$$

□

We define the scale-free adaptive statistic for testing exponentiality

$$T = \sum_{j\geq 0}\lambda_j\,\tilde{\Phi}_{nj} = \sum_{j\geq 0}\lambda_j\left(\frac{1}{n\,\bar{y}_n}\sum_{i=1}^n a_{ij}\,y_{(i)}\right) \tag{4}$$

where $\{\lambda_j\} \in \ell^1_{\mathbb{R}}$ is a sequence of real numbers.

In practice, given an alternative distribution F_1, we will use T_p, the result of truncating (4) at $j = p$, where parameters $\lambda_0, \dots, \lambda_p$ will be determined in order to maximize power for testing $H_0 : F = Exp(0, \beta)$ with unknown scale parameter $\beta > 0$ versus $H_1 : F = F_1$. Once coefficients λ_j's are known, it is possible to obtain the exact distribution of T_p under the null hypothesis (see Sect. 5) and hence to obtain the exact critical regions. It is also interesting to mention that for practical purposes T_p should be expressed directly in terms of the observed order statistic.

3 Optimization of the power function

Let $\mathbf{x} = (x_{(1)}, \dots, x_{(n)})'$ be the order statistic of n iid$\sim Exp(0, 1)$ random variables and consider the following auxiliary parametric function

$$\mathcal{L}^r_{np} = \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{j=0}^{p} a_{ij}\, \lambda_j - r \right) x_{(i)}, \tag{5}$$

where $a_{ij} = n \int_{(i-1)/n}^{i/n} \phi_j(t)\, dt$, $j = 0, \dots, p$, and $r, \lambda_0, \dots, \lambda_p$ are real coefficients to be determined. Notice that $\mathcal{L}^r_{np} = (T_p - r)\, \overline{x}_n$.

$\mathcal{L}^r_{np}$ is a linear combination of the order statistic and, in matrix notation, it can be written as $\mathcal{L}^r_{np} = \mathbf{L}'\, \mathbf{A}'\, \mathbf{x}$, where $\mathbf{L} = (r, \lambda_0, \dots, \lambda_p)'$ and

$$\mathbf{A} = \frac{1}{n} \begin{pmatrix} -1 & a_{10} & \dots & a_{1p} \\ -1 & a_{20} & \dots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ -1 & a_{n0} & \dots & a_{np} \end{pmatrix}.$$

Its expectation and variance are given by $\mu_i = \mathbf{L}'\, \mathbf{A}'\, \mathbf{m}_i$, $\sigma_i^2 = \mathbf{L}'\, \mathbf{A}'\, \Sigma_i\, \mathbf{A}\, \mathbf{L}$, where $\mathbf{m}_i = E(\mathbf{x}|H_i)$, $\Sigma_i = Var(\mathbf{x}|H_i)$, $i = 0, 1$, and applying the general theory of L-statistics, it is asymptotically normal distributed (see, for example, Shorack and Wellner 1986).

In order to find the coefficients $\lambda_0, \dots, \lambda_p$ such that the statistic $T_p = \sum_{j=0}^{p} \lambda_j\, \tilde{\Phi}_{nj}$ has maximum power to test $H_0 : F = Exp(0, \beta)$ with unknown $\beta > 0$ versus $H_1 : F = F_1$, we impose the following conditions, for a given significance level $\varepsilon \in (0, 1)$:

$$P(T_p > r|H_0) = \varepsilon, \quad P(T_p > r|H_1) \text{ is maximum.}$$

Using the auxiliary parametric function introduced in (5) these conditions are written:

$$P(\mathcal{L}^r_{np} > 0|H_0) = \varepsilon \quad \Rightarrow \quad \mathbf{L}'\, \mathbf{A}'\, \mathbf{m}_0 = -c_\varepsilon (\mathbf{L}'\, \mathbf{A}'\, \Sigma_0\, \mathbf{A}\, \mathbf{L})^{1/2}, \tag{6}$$

where c_ε is the $(1 - \varepsilon) \times 100$ percentile of the standard normal distribution, and the probability to be maximized can be asymptotically approximated by

$$\Psi(\mathbf{L}) = 1 - F_Z \left(\frac{-\mathbf{L}'\,\mathbf{A}'\,\mathbf{m}_1}{(\mathbf{L}'\,\mathbf{A}'\,\Sigma_1\,\mathbf{A}\,\mathbf{L})^{1/2}} \right) \tag{7}$$

where F_Z is the standard normal distribution function. To maximize the asymptotic power function is equivalent to find the extremes of the following quotient of quadratic forms:

$$\frac{\mathbf{L}'\,\mathbf{A}'\,\mathbf{m}_1\,\mathbf{m}_1'\,\mathbf{A}\,\mathbf{L}}{\mathbf{L}'\,\mathbf{A}'\,\Sigma_1\,\mathbf{A}\,\mathbf{L}}, \tag{8}$$

which is also equivalent to find the extremes of $\mathbf{L}'\,\mathbf{A}'\,\mathbf{m}_1\,\mathbf{m}_1'\,\mathbf{A}\,\mathbf{L}$ constrained to $\mathbf{L}'\,\mathbf{A}'\,\Sigma_1\,\mathbf{A}\,\mathbf{L} = 1$. Since the numerator of (8) is a matrix of rank one, the solution of (7) is the (unique with non-null eigenvalue) eigenvector of the generalized eigenvalue problem: $\mathbf{A}'\,\mathbf{m}_1\,\mathbf{m}_1'\,\mathbf{A}\,\mathbf{L} = \xi\,\mathbf{A}'\,\Sigma_1\,\mathbf{A}\,\mathbf{L}$, normalized so that $\mathbf{L}'\,\mathbf{A}'\,\Sigma_1\,\mathbf{A}\,\mathbf{L} = 1$, whenever $\mathbf{A}'\,\Sigma_1\,\mathbf{A}$ is positive defined, that is:

$$\mathbf{L} = (\mathbf{A}'\,\Sigma_1\,\mathbf{A})^{-1}\,\mathbf{A}'\,\mathbf{m}_1, \quad \xi = \mathbf{m}_1'\,\mathbf{A}\,(\mathbf{A}'\,\Sigma_1\,\mathbf{A})^{-1}\,\mathbf{A}'\,\mathbf{m}_1.$$

Observe that when the first element of the orthonormal basis $\{\phi_j(t)\}_{j\geq 0}$ is equal to 1, that is $\phi_0(t) = 1$, then $\mathbf{A}'\,\Sigma_1\,\mathbf{A}$ will be positive semidefined, since the first two columns of matrix $\mathbf{A}$ will be proportional. In this case, the solution of (7) is

$$\mathbf{L} = \mathbf{B}^-\,\mathbf{A}'\,\mathbf{m}_1 + \mathbf{N}\,\mathbf{h}, \quad \xi = \mathbf{m}_1'\,\mathbf{A}\,\mathbf{B}^-\,\mathbf{A}'\,\mathbf{m}_1,$$

where $\mathbf{B}^-$ is the Moore–Penrose pseudoinverse of $\mathbf{B} = \mathbf{A}'\,\Sigma_1\,\mathbf{A}$, $\mathbf{N} = \mathbf{I} - \mathbf{B}\,\mathbf{B}^-$ and $\mathbf{h}$ is a $(p+2)\times 1$ arbitrary vector (see McDonald et al. 1979). Condition (6) helps to determine vector $\mathbf{h}$.

Note that to solve this problem it is necessary that all the first and second order moments of the order statistic involved do exist. This could be not the case when the expectation of the distribution does not exist due to singularities in 0 or 1 (see David 1981).

4 Generic alternatives

In this section we find approximations of the expectation and variance (under H_1) of the parametric function defined in (5), in order to compute the optimal vector $\mathbf{L}$ of Sect. 3. We will suppose that the alternative probability distribution function F has a pseudoinverse of the form:

$$F^-(t) = \sum_{k=0}^{q} \gamma_k\,\psi_k(t), \tag{9}$$

where γ_k are real numbers and $\{\psi_k(t)\}_{k\geq 0}$ is an orthonormal sequence in $L^2[0, 1]$, possibly different from $\{\phi_j(t)\}_{j\geq 0}$.

Given an arbitrary F the first q Fourier terms of F^- yield such an expression. In the present context this is more natural than expanding F or the probability density function, since the moments of the order statistic can be advantageously expressed in terms of F^{-}, e.g.,

$$E(x_{(i)}|H_1) = i\binom{n}{i}\int_0^1 F^{-}(t)\, t^{i-1}\,(1-t)^{n-i}\,dt$$
$$= i\binom{n}{i}\sum_{k=0}^{q}\gamma_k\int_0^1 \psi_k(t)\, t^{i-1}\,(1-t)^{n-i}\,dt. \tag{10}$$

To solve (7) we must determine $\mathbf{m}_1$ and Σ_1. Formula (10) gives the entries in $\mathbf{m}_1$, but for several reasons, an exact Σ_1 could not be available. Instead we can determine $\mathbf{A}'\Sigma_1\mathbf{A}$ from the asymptotic approximation given in the following proposition.

Proposition 4.1 *Let $\{\phi_j(t)\}_{j\geq 0}$ and $\{\psi_k(t)\}_{k\geq 0}$ be two orthonormal systems in $L^2[0,1]$ such that $\phi_0(t)=\psi_0(t)=1$, $\mathcal{L}^r_{np}$ the parametric function defined in (5) and $(\ell_1,\ell_2,\ldots,\ell_{p+2})=(r,\lambda_0,\ldots,\lambda_p)$.*

We have the following convergences in law

$$\sqrt{n}\left[\mathcal{L}^r_{np}-\mu_{\mathcal{L}}\right]\xrightarrow[n\to\infty]{\mathcal{L}} N(0,\sigma^2_{\mathcal{L}}), \tag{11}$$

$$\sqrt{n}\frac{\left[\mathcal{L}^r_{np}-\mu_{\mathcal{L}}\right]}{\sigma_n}\xrightarrow[n\to\infty]{\mathcal{L}} N(0,1), \tag{12}$$

where

$$\mu_{\mathcal{L}} = (-\ell_1+\ell_2)\,\gamma_0+\sum_{j=1}^{p}\sum_{k=1}^{q}\ell_{j+2}\,\gamma_k\int_0^1\phi_j(t)\,\psi_k(t)\,dt \tag{13}$$

is the asymptotic expectation of $\mathcal{L}^r_{np}$ and $\sigma^2_{\mathcal{L}}=\lim_{n\to\infty}\sigma^2_n$, where

$$\sigma^2_n = n\ \mathrm{var}(\mathcal{L}^r_{np}) = \frac{1}{n}\sum_{j=1}^{p+2}\sum_{l=1}^{p+2}\ell_j\,\ell_l\,\sigma_{jl},\quad \sigma_{jl}=\sum_{k=0}^{q}\sum_{m=0}^{q}\gamma_k\,\gamma_m\,I_{jklm}, \tag{14}$$

where

$$I_{jklm} = \begin{cases} \int_0^1\int_0^1 K(s,t)\,\psi'_k(s)\,\psi'_m(t)\,ds\,dt, & j=l=1,\\ \int_0^1\int_0^1 K(s,t)\,\phi_{l-2}(t)\,\psi'_k(s)\,\psi'_m(t)\,ds\,dt, & j=1, l\geq 2,\\ \int_0^1\int_0^1 K(s,t)\,\phi_{j-2}(t)\,\psi'_k(s)\,\psi'_m(t)\,ds\,dt, & l=1, j\geq 2,\\ \int_0^1\int_0^1 K(s,t)\,\phi_{j-2}(s)\,\psi'_k(s)\,\phi_{l-2}(t)\,\psi'_m(t)\,ds\,dt, & j\geq 2, l\geq 2,\end{cases}$$

with $K(s,t)=\min(s,t)-s\,t$ and $\psi'_k(t)$ denotes the derivative of $\psi_k(t)$.

Proof From the asymptotic theory of L-statistics (see Chap. 19 of Shorack and Wellner 1986, for the notation and the constructions used), the parametric function of (5) can be written as

$$\mathcal{L}_{np}^r = \int_0^1 J_n^r(t)\, F_n^-(t)\, dt,$$

where F_n^- is the pseudo-inverse of the empirical distribution and

$$J_n^r(t) = -B_0(t)\,\ell_1 + \sum_{j=0}^{p} B_j(t)\,\ell_{j+2},$$

is a bounded and continuous a.e. (F^-) function, where $(\ell_1, \ell_2, \ldots, \ell_{p+2}) = (r, \lambda_0, \ldots, \lambda_p)$, $B_j(t) = \left(b_j(t) - b_j(t - 1/n)\right)/(1/n)$ and $b_j(s) = \int_0^s \phi_j(t)\, dt$, for $j = 0, \ldots, p$.

Noticing that $\lim_{n\to\infty} B_j(t) = \phi_j(t)$ and taking into account Remark 2 of Stigler (1974), we can define the following limit function that will be used to compute the asymptotic expectation and variance of $\mathcal{L}_{np}^r$,

$$J(t) = \lim_{n\to\infty} J_n^r(t) = -\phi_0(t)\,\ell_1 + \sum_{j=0}^{p} \phi_j(t)\,\ell_{j+2}, \quad t \in (0, 1),$$

which is also a continuous and bounded a.s. (F^-) function. Hence,

$$\begin{aligned}
\mu_{\mathcal{L}} &= \int_0^1 J(t)\, F^-(t)\, dt = \int_0^1 \left(-\phi_0(t)\,\ell_1 + \sum_{j=0}^{p} \phi_j(t)\,\ell_{j+2}\right) \sum_{k=0}^{q} \gamma_k\, \psi_k(t)\, dt \\
&= (-\ell_1 + \ell_2)\,\gamma_0 + \sum_{j=1}^{p}\sum_{k=1}^{q} \ell_{j+2}\,\gamma_k \int_0^1 \phi_j(t)\,\psi_k(t)\, dt,
\end{aligned}$$

where we have used that $\phi_0(t) = \psi_0(t) = 1$ and $\int_0^1 \phi_j(t)\, dt = \int_0^1 \psi_k(t)\, dt = 0$ for $j \geq 1$ and $k \geq 1$. The asymptotic variance is given by

$$\sigma_{\mathcal{L}}^2 = \int_0^1\int_0^1 J(s)\, J(t)\, K(s, t)\, dF^-(s)\, dF^-(t),$$

where $K(s, t) = \min(s, t) - st$. Formula (14) is obtained substituting the expressions for function J and for the derivative of F^- and proceeding analogously as for the computation of the asymptotic expectation.

The convergence of (11) is obtained from Theorem 1 of Shorack and Wellner (1986, pp. 664–665). The convergence (12) is immediate from (11) and from the fact that $\sigma^2_{\mathcal{L}} = \lim_{n\to\infty} \sigma^2_n$. □

As a final comment to this section, note that the orthonormal system $\{\psi_k\}_{k\geq 0}$ should be chosen carefully so that a good approximation for F^- is guaranteed. In Grané and Fortiana (2008) we use the Bahadur asymptotic relative efficiency as a helpful quantity in choosing an adequate basis.

5 Exact distribution of T_p under H_0

Proposition 5.1 *Given $\lambda_0, \ldots, \lambda_p \in \mathbb{R}$, the statistic $T_p = \sum_{j=0}^p \lambda_j\, \tilde{\Phi}_{nj}$ does not depend on the scale parameter, and can be written as*

$$T_p = \frac{\sum_{i=1}^n c_i\, z_i}{\sum_{i=1}^n z_i},$$

where $c_i = \sum_{k=i}^n \frac{1}{n-i+1} \sum_{j=0}^p a_{kj}\, \lambda_j$, $i = 1, \ldots, n$, $a_{kj} = n \int_{(k-1)/n}^{k/n} \phi_j(t)\, dt$, $j = 1, \ldots, p$ and $z_1, \ldots, z_n$ are iid$\sim$ $Exp(0, 1)$ random variables.

Proof Suppose $x_j = y_j/\beta$, $j = 1, \ldots, n$, then after truncating formula (4) at $j = p$ we get:

$$T_p = \sum_{j=0}^p \lambda_j \left(\frac{\sum_{i=1}^n a_{ij}\, x_{(i)}}{\sum_{i=1}^n x_{(i)}} \right) = \frac{\sum_{i=1}^n b_i\, x_{(i)}}{\sum_{i=1}^n x_{(i)}}, \tag{15}$$

where $b_i = \sum_{j=0}^p \lambda_j\, a_{ij}$, $i = 1, \ldots, n$. From David (1981), for $i = 1, \ldots, n$, the transformation $z_i = (n - i + 1)(x_{(i)} - x_{(i-1)})$, where, by convention, $x_{(0)} = 0$, gives n iid$\sim$ $Exp(0, 1)$ random variables, and the i-th order statistic from the standard distribution can be written as

$$x_{(i)} = \sum_{k=1}^i \frac{z_k}{n - k + 1} \tag{16}$$

Substituting (16) in (15),

$$T_p = \frac{\sum_{i=1}^n \sum_{k=1}^i b_i \frac{z_k}{(n-k+1)}}{\sum_{i=1}^n \sum_{k=1}^i \frac{z_k}{(n-k+1)}} = \frac{\sum_{i=1}^n \frac{z_i}{(n-i+1)} \sum_{k=i}^n b_k}{\sum_{i=1}^n z_i} = \frac{\sum_{i=1}^n c_i\, z_i}{\sum_{i=1}^n z_i},$$

where $c_i = \sum_{k=i}^n \frac{b_k}{n-i+1} = \sum_{k=i}^n \left(\frac{1}{n-i+1} \sum_{j=0}^p a_{kj}\, \lambda_j \right)$, $i = 1, \ldots, n$. □

From now on we will suppose that coefficients c_i are non-null and that $c_i \neq c_j$ for all $i \neq j$, in order to simplify notation and also because this is exactly what happens in the applications considered in Sect. 6. Let $N = \sum_{i=1}^{n} c_i\, z_i$ and $D = \sum_{i=1}^{n} z_i$ so that $T_p = N/D$.

Lemma 5.1 *The characteristic function of the vector* (N, D) *is*

$$\varphi_{(N,D)}(t_1, t_2) = \prod_{j=1}^{n} \left(1 - i(c_j\, t_1 + t_2)\right)^{-1},$$

with $(t_1, t_2) \in \mathbb{R}^2$, $i = \sqrt{-1}$.

Proof In matrix notation, $N = \mathbf{c}'\,\mathbf{z}$, $D = \mathbf{1}'\,\mathbf{z}$, where $\mathbf{c} = (c_1, \ldots, c_n)'$, $\mathbf{z} = (z_1, \ldots, z_n)'$ and $\mathbf{1} = (1, \ldots, 1)'$. We also consider the matrix $\mathbf{C} = (\mathbf{c}, \mathbf{1})'$. Then,

$$\begin{pmatrix} N \\ D \end{pmatrix} = \mathbf{C}\,\mathbf{z}.$$

From the transformation formula of characteristic functions by the action of an affine map, we have that $\varphi_{\mathbf{Cz}}(\mathbf{t}) = \varphi_{\mathbf{z}}(\mathbf{C}'\,\mathbf{t})$, for $\mathbf{t} = (t_1, t_2)' \in \mathbb{R}^2$. Since $z_1, \ldots, z_n$ are iid$\sim$ $Exp(0, 1)$ random variables,

$$\varphi_{\mathbf{z}}(\mathbf{C}'\,\mathbf{t}) = \prod_{j=1}^{n} \varphi_{z_j}((c_j\, t_1, t_2)) = \prod_{j=1}^{n} \left(1 - i(c_j\, t_1 + t_2)\right)^{-1}.$$

□

Since T_p is a quotient of linear combinations of iid$\sim$ $Exp(0, 1)$ random variables, to obtain its exact probability density function we can adapt the technique used in Dwass (1961) and Matsunawa (1985).

Proposition 5.2 *The exact probability density function of* T_p *is given by*

$$f(t) = (n-1) \sum_{l=1}^{n} sgn(c_l) \prod_{k=1, k\neq l}^{n} (c_l - c_k)^{-1}\, (c_l - t)^{n-2}\, \chi(t/c_l)\, \chi(1 - t/c_l),$$

where $\chi(s)$ *is the indicator of the interval* $[s > 0]$.

Proof using Lemma 2.2 from Matsunawa (1985), we obtain the characteristic function of (N, D):

$$\varphi_{(N,D)}(t_1, t_2) = \prod_{j=1}^{n} c_j^{-1} \sum_{l=1}^{n} e_l\, c_l\, \left((1 - i\, t_2) - i\, c_l\, t_1\right)^{-1} (1 - i\, t_2)^{1-n},$$

where

$$e_l = \prod_{k=1, k\neq l}^{n} \left(\frac{1}{c_k} - \frac{1}{c_l}\right)^{-1}.$$

Inverting this characteristic function, following formula 2.1 in Matsunawa (1985):

$$f_{(N,D)}(w,s) = \sum_{l=1}^{n} e_l \left(\prod_{j=1}^{n} c_j^{-1} \right) \frac{\exp(-s)}{\Gamma(n-1)} \left(s - \frac{w}{c_l} \right)^{n-2} \chi\left(\frac{w}{c_l}\right) \chi\left(s - \frac{w}{c_l}\right),$$

where $\chi(s)$ is the indicator of the interval $[s > 0]$, for $(w, s) \in \mathbb{R}^2$. Applying a change of variables and noticing that

$$e_l \left(\prod_{j=1}^{n} c_j^{-1} \right) = c_l^{n-2} \prod_{k=1, k \neq l} (c_l - c_k)^{-1},$$

we get the probability density function of T_p, $f(t) = \int_{-\infty}^{+\infty} |s|\, f_{(N,D)}(ts, s)\, ds$. □

We have developed a *Mathematica* program implementing this algorithm, which also computes the exact critical values for a given significance level.

6 Power of T_p

In this section we study the power of the (one-sided) test of exponentiality based on T_p designed to distinguish a fixed alternative distribution and we compare it with a large number of recent and classical tests of exponentiality.

6.1 Alternatives to exponentiality

We have considered families of the form $F(x; \theta_1, \theta_2)$. In each case, one of the θ_1, θ_2 is the scale parameter and can be fixed to a given arbitrary value, without loss of generality.

A1. The first family of alternatives is the Weibull distribution with shape parameter α and scale parameter β, whose density function is:

$$f(x; \alpha, \beta) = \frac{\alpha}{\beta^\alpha} x^{\alpha-1} \exp\{-(x/\beta)^\alpha\}, \quad \alpha, \beta > 0,\ x \geq 0.$$

Note that $\alpha = 1$ corresponds to H_0 and that the Weibull distribution is more similar to the exponential distribution when $\alpha < 1$.

A2. The second family of alternatives is the Generalized Pareto, whose distribution function is:

$$F(x; a, k) = 1 - \left(1 - \frac{k}{a} x\right)^{1/k}, \quad a > 0,$$

with $0 \leq x < \infty$, if $k \leq 0$, and $0 \leq x \leq a/k$ if $k > 0$. Observe that $\lim_{k \to 0} F(x; a, k) = Exp(0, a)$, and that the Generalized Pareto distribution is similar to the exponential distribution only when $k < 1$.

As examples of construction of the test statistic T_p and motivated by the reasons explained above, we have considered the A1 and A2 alternatives for some specific values of the parameters, for which it is also possible to compute the exact values of $\mathbf{m}_1$ and Σ_1 of Sect. 3. For a sample size of $n = 10$ and a 5% significance level we have determined coefficients $\lambda_0, \ldots, \lambda_p$ for $p = 4$ using the orthonormal basis obtained from the sequence of Legendre polynomials $\phi_0(t) = 1$, $\phi_1(t) = \sqrt{3}(2t - 1)$, and recurrence relation:

$$\phi_{j+1}(t) = \frac{\sqrt{(2j+3)(2j+1)}}{j+1}(2t-1)\,\phi_j(t) - \frac{\sqrt{2j+3}}{\sqrt{2j-1}}\frac{j}{j+1}\,\phi_{j-1}(t), \quad j \geq 1.$$

Applying Proposition 5.2 we have computed the exact critical region of the test.

Concerning the orthonormal basis, there are other orthonormal systems that could be used in the construction of T_p. Here we have chosen the Legendre polynomials, mainly for simplicity, and also because they are often used in the context of *smooth* tests [see, Inglot et al. (1994) and Kallenberg and Ledwina (1997) for a discussion on how to choose the number of p components and how to select an appropriate orthonormal system]. In Grané and Fortiana (2000) we considered the orthonormal system obtained from the eigenfunctions of the covariance kernel of a certain stochastic process associated with the exponential distribution. Another possibility in the selection of the orthonormal basis would be to proceed as in Grané and Fortiana (2008).

6.2 Test statistics

For the power comparisons, we have chosen a representative number of the rich class of tests discussed in Henze and Meintanis (2005). Following their notation, we will denote the scaled observations as $y_i = x_i/\bar{x}$. Tables 1 and 2 contain the power comparisons of the 5%-significance level tests based on:

1. The Gini statistic (Gail and Gastwirth 1978):

$$G = \frac{1}{n(n-1)} \sum_{i=1}^{n} (2i - n - 1)\, y_{(i)}.$$

2. The statistic of Cox and Oakes (1984)

$$CO_n = n + \sum_{i=1}^{n} (1 - y_i)\, \log(y_i).$$

3. The statistic of Epps and Pulley (1986):

$$EP_n = \sqrt{48n}\left(\frac{1}{n}\sum_{i=1}^{n} \exp(-y_i) - \frac{1}{2}\right)$$

Table 1 Power comparisons of the tests of exponentiality versus A1 and A2 alternatives, for $n = 10$ and a 5% significance level

α	T_p	Q_n	W_S	G	KL	EP	CO	$\overline{KS}$	$\overline{CM}$	HE	$HM^{(1)}$	$HM^{(2)}$
A1 family ($\beta = 1$)												
0.25	0.998	0.955	0.956	0.980	0.802	0.982	1.000	0.932	0.972	0.969	0.955	0.927
0.5	0.789	0.641	0.653	0.768	0.125	0.681	0.910	0.452	0.599	0.645	0.566	0.601
1.5	0.345	0.326	0.339	0.353	0.328	0.888	0.294	0.262	0.219	0.236	0.234	0.017
2	0.735	0.715	0.765	0.755	0.712	0.995	0.629	0.615	0.599	0.624	0.562	0.058
3	0.993	0.987	0.991	0.994	0.981	1.000	0.988	0.942	0.962	0.989	0.970	0.401
k	T_p	Q_n	W_S	G	KL	EP	CO	$\overline{KS}$	$\overline{CM}$	HE	$HM^{(1)}$	$HM^{(2)}$
A2 family ($a = 1$)												
−0.2	0.161	0.154	0.126	0.152	0.035	0.214	0.244	0.056	0.106	0.096	0.139	0.175
0.25	0.126	0.111	0.115	0.109	0.115	0.438	0.048	0.084	0.057	0.061	0.073	0.010
0.5	0.235	0.207	0.211	0.208	0.202	0.599	0.039	0.171	0.125	0.129	0.158	0.010

Table 2 Power comparisons of the Neyman smooth tests for testing exponentiality versus A1 and A2 alternatives, for $n = 10$ and a 5% significance level

α	T_p	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
A1 family ($\beta = 1$)										
0.25	0.998	0.964	0.985	0.972	0.982	0.980	0.985	0.982	0.981	0.984
0.5	0.789	0.658	0.667	0.701	0.671	0.633	0.646	0.622	0.637	0.646
1.5	0.345	0.199	0.214	0.160	0.116	0.072	0.055	0.03	0.034	0.029
2	0.735	0.589	0.619	0.510	0.404	0.305	0.229	0.173	0.140	0.130
3	0.993	0.969	0.977	0.960	0.899	0.843	0.754	0.662	0.639	0.618
k	T_p	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
A2 family ($a = 1$)										
−0.2	0.161	0.127	0.136	0.111	0.109	0.114	0.113	0.105	0.089	0.095
0.25	0.126	0.061	0.061	0.044	0.028	0.026	0.017	0.013	0.011	0.009
0.5	0.235	0.124	0.132	0.095	0.062	0.034	0.021	0.025	0.026	0.020

4. The statistic proposed by Ebrahimi et al. (1992) based on the entropy estimator of Vasicek (1976)

$$KL_{mn} = \frac{\exp(H_{mn})}{\exp(\ln \overline{x} + 1)},$$

where $H_{mn} = \frac{1}{n}\sum_{i=1}^{n} \ln\left\{\frac{n}{2m}(x_{(i+m)} - x_{(i-m)})\right\}$, $x_{(j)} = x_{(1)}$ if $j < 1$, $x_{(j)} = x_{(n)}$ if $j > n$ and the window size m is a positive integer smaller than $n/2$. In this power study, we have taken $m = 3$, following the recommendations of Ebrahimi et al. (1992).

5. The statistic of Henze (1993) based on the Laplace transform:

$$HE_n = n \int_0^\infty \left(\psi_n(t) - \frac{1}{1+t}\right)^2 \exp(-a\,t)\,dt,$$

where $\psi_n(t) = n^{-1}\sum_{i=1}^n \exp(-t\,y_i)$ is the empirical Laplace transform of the unit exponential distribution. In this power study we have taken $a = 2.5$, following the recommendations of Henze and Meintanis (2005).

6. The Kolmogorov–Smirnov and Cramér–von Mises type statistics suggested by Baringhaus and Henze (2000)

$$\overline{KS}_n = \sqrt{n}\,\sup_{t\geq 0}\left|\frac{1}{n}\sum_{i=1}^n \min(y_i, t) - \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{y_i \leq t\}}\right|,$$

$$\overline{CM}_n = n \int_0^\infty \left(\frac{1}{n}\sum_{i=1}^n \min(y_i, t) - \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{y_i \leq t\}}\right)^2 \exp(-t)\,dt,$$

where $\mathbf{1}_A$ is the indicator of the event A, which is 1 if A occurs and 0 otherwise.

7. The statistics of Henze and Meintanis (2002) based on the empirical characteristic function:

$$HM_n^{(1)} = n \int_{-\infty}^{+\infty} (s_n(t) - t\,c_n(t))^2 \exp(-a\,t)\,dt,$$

$$HM_n^{(2)} = n \int_{-\infty}^{+\infty} (s_n(t) - t\,c_n(t))^2 \exp(-a\,t^2)\,dt,$$

where $c_n(\cdot)$ and $s_n(\cdot)$ are the real and imaginary part, respectively, of the empirical characteristic function $\phi_n(t) = n^{-1}\sum_{j=1}^n \exp(i\,t\,y_j)$. In this power study we have taken $a = 2.5$, following the recommendations of Henze and Meintanis (2005).

We have also included in the power comparisons of Table 1 the performance of the tests based on the following scale-free statistics:

8. Neyman smooth tests for testing exponentiality defined by (in the notation of Rayner and Best 1989):

$$S_{k-1} = \sum_{r=2}^k U_r^2,$$

Table 3 Coefficients of the T_p statistic ($p = 4$) for the one-sided test to detect departures from the exponentiality, for a sample size of $n = 10$ and a 2.5% significance level

CV	T_p coefficients					critical region
$CV > 1$	0.250882	−0.185657	0.065586	−0.025221	0.002454	$(-\infty, 0.086114)$
$CV \leq 1$	0.500807	0.440998	−0.232712	0.406716	−0.321763	$(-\infty, 0.715806)$

CV stands for coefficient of variation

where $U_r = \sum_{j=1}^{n} \pi_r(y_j)/\sqrt{n}$, with $\{\pi_r, r \geq 1\}$ the sequence of Laguerre polynomials $\pi_1(y) = 1-y, \pi_2(y) = 1-2\,y+y^2/2$ and recurrence relation $r\,\pi_r(y) = (2\,r-1-y)\,\pi_{r-1}(y) - (r-1)\pi_{r-2}(y)$, for $r > 2$.

9. The Stephens' modification of the Shapiro–Wilk statistic Stephens (1978):

$$W_s = \frac{(\sum_{i=1}^{n} x_i)^2}{n(n+1)\sum_{i=1}^{n} x_i^2 - n(\sum_{i=1}^{n} x_i)^2}.$$

This statistic coincides, up to one-to-one transformations, with the Greenwood's Greenwood (1946) statistic and with the first nonzero component of Neyman's smooth test for testing exponentiality, based on the second Laguerre polynomial, that is $S_1 = U_2^2$, (see Rayner and Best 1989 and Henze and Klar 1996).

10. The Q_n statistic defined in Fortiana and Grané (2002):

$$Q_n = \frac{\sum_{i=1}^{n} l_i\, x_{(i)}}{n\,\bar{x}},$$

where $l_i = (n-i)\log(n-i) - (n-i+1)\log(n-i+1) + \log(n), i = 1, \ldots, n$ and $0\,\log 0 \equiv 0$.

For all the statistics involved, the powers have been estimated from $N = 1{,}000$ simulated samples of size $n = 10$ as the relative frequency of values of the corresponding statistic in the critical region. Table 1 contains the comparison of the power of the test based on T_p with those based on Q_n, W_S, G, KL, EP, CO, $\overline{KS}$, $\overline{CM}$, HE, $HM^{(1)}$ and $HM^{(2)}$. As a general comment, it can be said that there is no leading statistic, although EP, G and T_p statistics are in the first three positions for practically all the scenarios considered. Table 2 contains the comparison of the power of the test based on T_p with those based on Neyman smooth tests (S_2, S_3,..., S_{10}). This second table reinforces the good performance of T_p for the alternatives considered. The principal drawback of T_p is that its coefficients $(\lambda_0, \ldots, \lambda_p)$ should be computed anew for each parameter value. Although this procedure can be automated, we would like to find a set of coefficients that make T_p optimal enough for very practical purposes. This is the reason why in the next section we investigate whether and when the test based on T_p (that has been designed to distinguish a particular alternative) performs well in detecting other departures from the exponentiality.

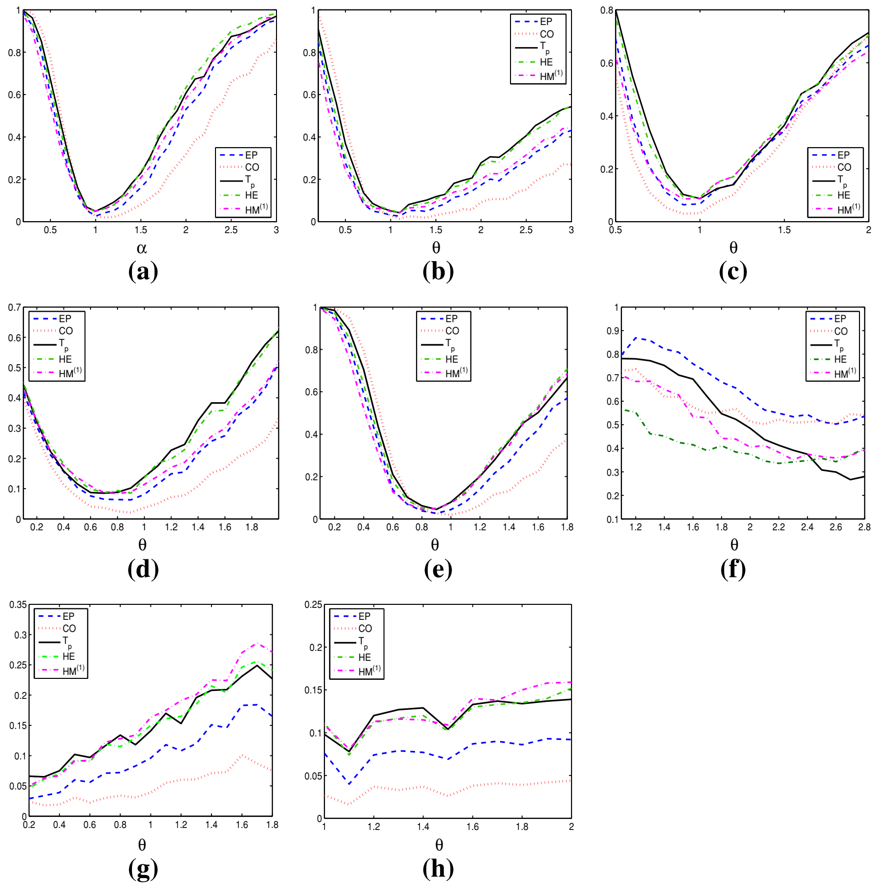

Fig. 1 Comparison of the tests based on EP, CO, T_p, HE and HM for **a** Weibull distribution, **b** Gamma distribution, **c** lognormal distribution, **d** Dhillon's law, **e** Chen distribution, **f** shifted Pareto distribution, **g** modified extreme value distribution and **h** linear increasing failure rate law

7 Power study for several alternatives

Although the test based on T_p applies when we are interested in distinguishing the null hypothesis of exponentiality from a fixed alternative, in this section we are interested in studying the performance of this test in detecting other alternatives for which it has been not designed. We may remark that the test based in T_p is not adaptive in the usual sense, but it can be regarded as some kind of compromise between an omnibus test and a directional test.

Inspired by the works of Epps and Pulley (1986); Henze and Meintanis (2005) and Cabaña and Cabaña (2005) we have chosen the following families of distributions, defined either by its probability density function (pdf) or its cumulative distribution function (cdf):

- Weibull distribution with parameters $(\alpha, 1)$ (see Sect. 6.1),

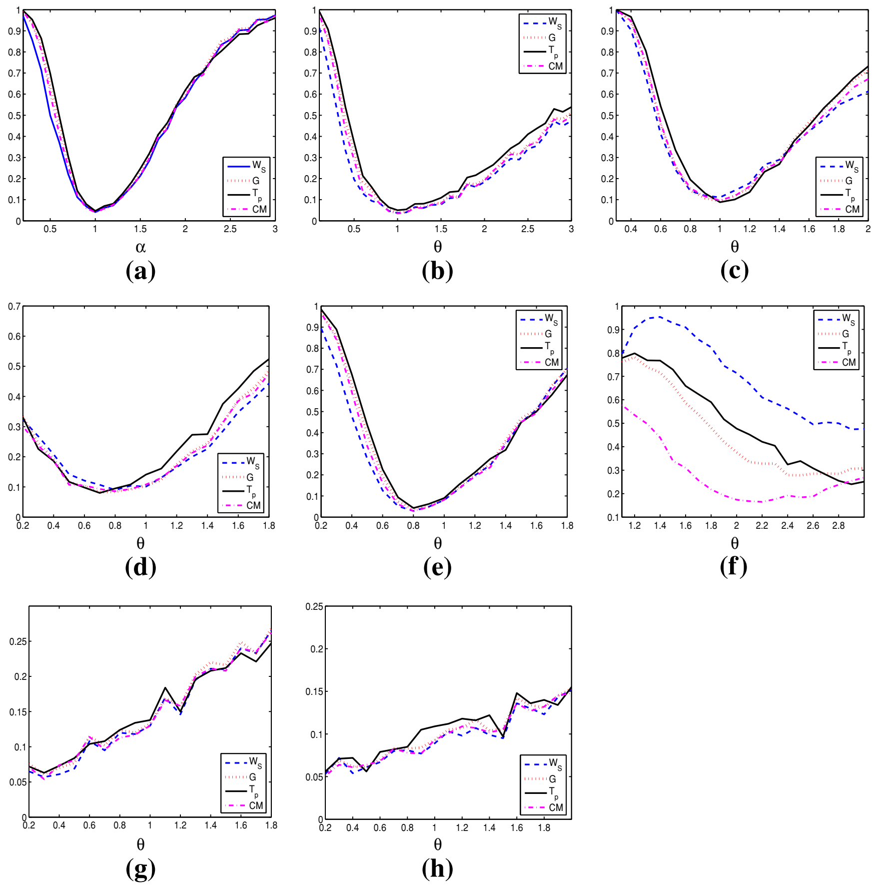

Fig. 2 Comparison of the tests based on W_S, G, T_p and $\overline{CM}$ for **a** Weibull distribution, **b** Gamma distribution, **c** lognormal distribution, **d** Dhillon's law, **e** Chen distribution, **f** shifted Pareto distribution, **g** modified extreme value distribution and **h** linear increasing failure rate law

- Gamma distribution with pdf $f(x;\theta) = \Gamma(\theta)^{-1}\, x^{\theta-1}\, e^{-x}$,
- Lognormal distribution with pdf $f(x;\theta) = \frac{1}{\theta\, x\sqrt{2\pi}} \exp\left(-(\log x)^2/(2\theta)^2\right)$,
- Modified Extreme Value distribution with cdf $F(x;\theta) = 1 - \exp\left(\theta^{-1}(1 - e^x)\right)$,
- Linear Increasing Failure Rate law with pdf $f(x;\theta) = (1 + \theta\, x)\exp(-x - \theta\, x^2/2)$,
- Shifted Pareto distribution with pdf $f(x;\theta) = (1 + x)^{-(1+\theta)}$,
- Dhillon's (Dhillon 1981) law with dcf $F(x;\theta) = 1 - \exp(-(\log(x + 1))^{\theta+1})$,
- Chen's (Chen 2000) law with cdf $F(x;\theta) = 1 - \exp(2(1 - e^{x^\theta}))$.

The T_p statistic under study is the one designed to distinguish the Weibull distribution (family A1 of Sect. 6.1). For this family we constructed the T_p statistic for several parameter values and we selected the set of coefficients $(\lambda_0, \ldots, \lambda_p)$ for which T_p was globally more powerful. These selected sets, which are shown in Table 3, correspond to values of $\alpha = 0.5$ and $\alpha = 2$. We have also observed that the statistic

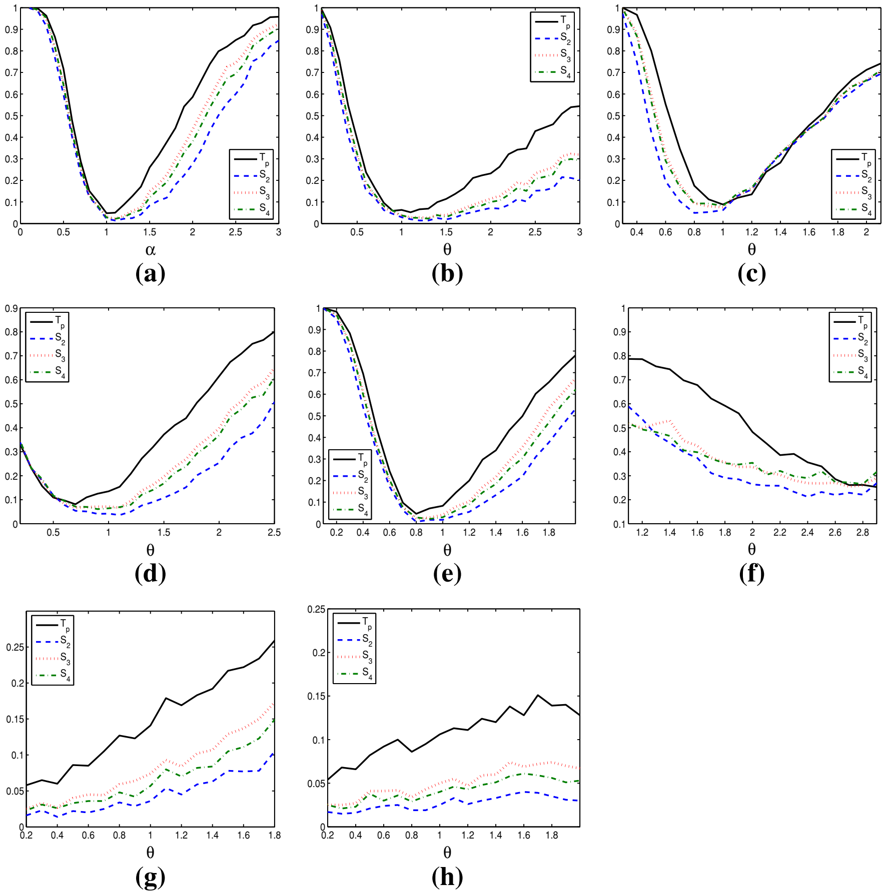

Fig. 3 Comparison of the tests based on S_2, S_3, S_4 and T_p for **a** Weibull distribution, **b** Gamma distribution, **c** lognormal distribution, **d** Dhillon's law, **e** Chen distribution, **f** shifted Pareto distribution, **g** modified extreme value distribution and **h** linear increasing failure rate law

constructed for $\alpha = 0.5$ has a good performance in detecting alternative distributions whose coefficient of variation is greater than 1, and that the opposite happens for the statistic constructed for $\alpha = 2$. Hence, in order to design a "globally optimal" test of significance level ε, we will consider the two one-sided tests of level $\varepsilon/2$ based on the T_p statistics obtained for $\alpha = 0.5$ and $\alpha = 2$. Table 3 contains the coefficients of these tests, as well as their critical regions, for a sample size of $n = 10$ and a 2.5% significance level.

Figures 1, 2 and 3 contain the power curves of the 5% significance level test of exponentiality based on a representative group of the statistics introduced in Sect. 6.2. Power against an alternative distribution has been estimated by the relative frequency of values of the corresponding statistic in the critical region for $N = 1,000$ simulated samples of size $n = 10$.

Concluding remarks

From Fig. 1 we can see that the tests based HE and T_p perform rather similar, except for the shifted pareto distribution (panel f), where T_p is more powerful than HE for almost all the parameter values. The power curves of the statistics compared in Fig. 2 differ only in panels (b–f) and (h), where we can appreciate the good results obtained with T_p, except for the shifted pareto distribution. Finally, the power comparisons in Fig. 3 reinforce the good performance of the test based on T_p. Through this simulation study we have seen that, in practically all the alternatives considered, and for a wide range of the parameter values, the test based on T_p is at least as good as the best tests of exponentiality in the statistical literature.

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